AN INTRODUCTION TO GAME THEORY

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Game Theory

- What is Game Theory?
  - Game theory is a branch of applied mathematics that is often used in the context of economics.
  - It studies ***strategic interactions*** between agents. In strategic games, agents choose strategies which will maximize their return, given the strategies the other agents choose.
- The essential feature is that it provides a *formal modelling approach* to social situations in which decision makers interact with other agents.
- *Behavioural economics* introduces psychology into the mix too

Schelling (2005)
Nobel Prize

Game Theoretic ideas

- What is a "game"?

Zero-sum vs Non-zero sum

Skills vs Chance

Sequential vs Simultaneous

Strategic vs Independent

Rational vs Irrational

First-mover advantage vs Second-mover advantage

Shopping Mall story

Pyrrhic victory

Cheating

Dominant vs Dominated strategies

Brinkmanship

Repeated vs One-shot games

Tipping Points

Pure vs Mixed strategies

Focal Points

Flat Tire story

Backward Induction

Rollback Analysis

Nash Equilibirum

Punishment strategies

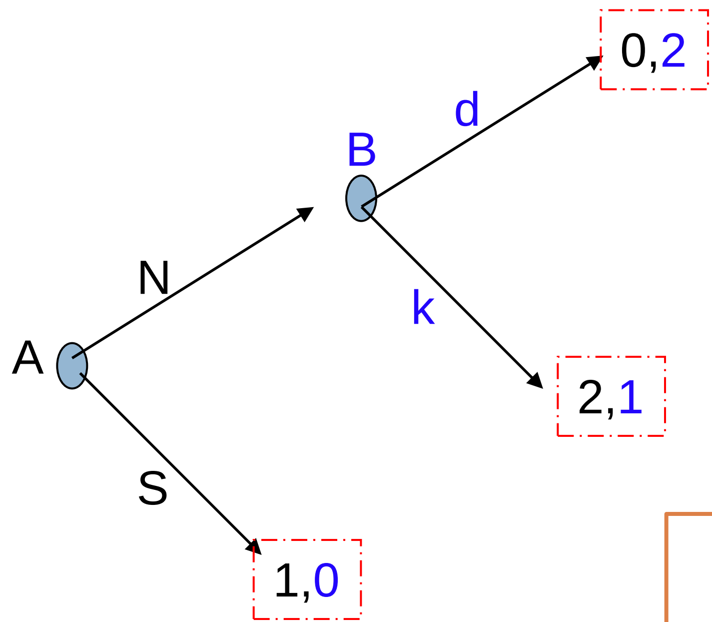
Backward Induction (Rollback analysis)

Extensive form

Game Tree

Branches

Nodes



Start at the final node and work out the optimal response there…

Use Backward Induction

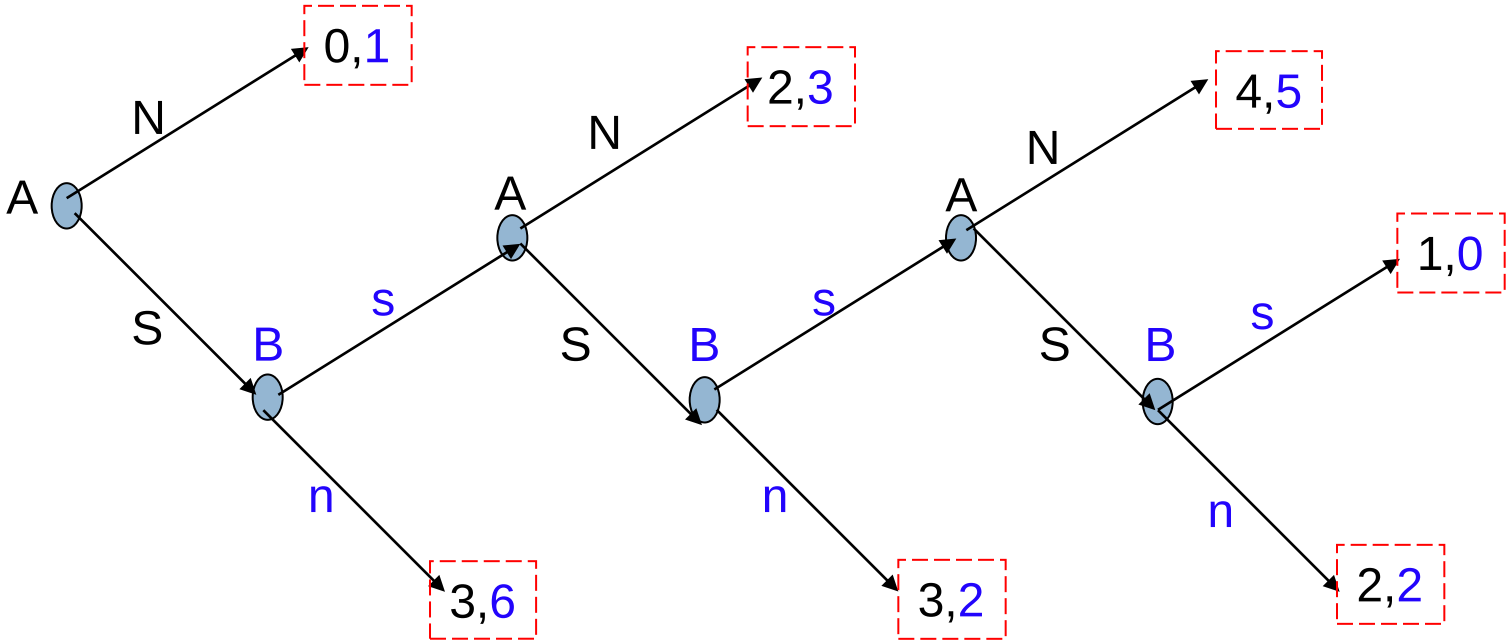
What is the Nash Equilibrium?

- Backward-induction applications
  - E.g. Apple; your career

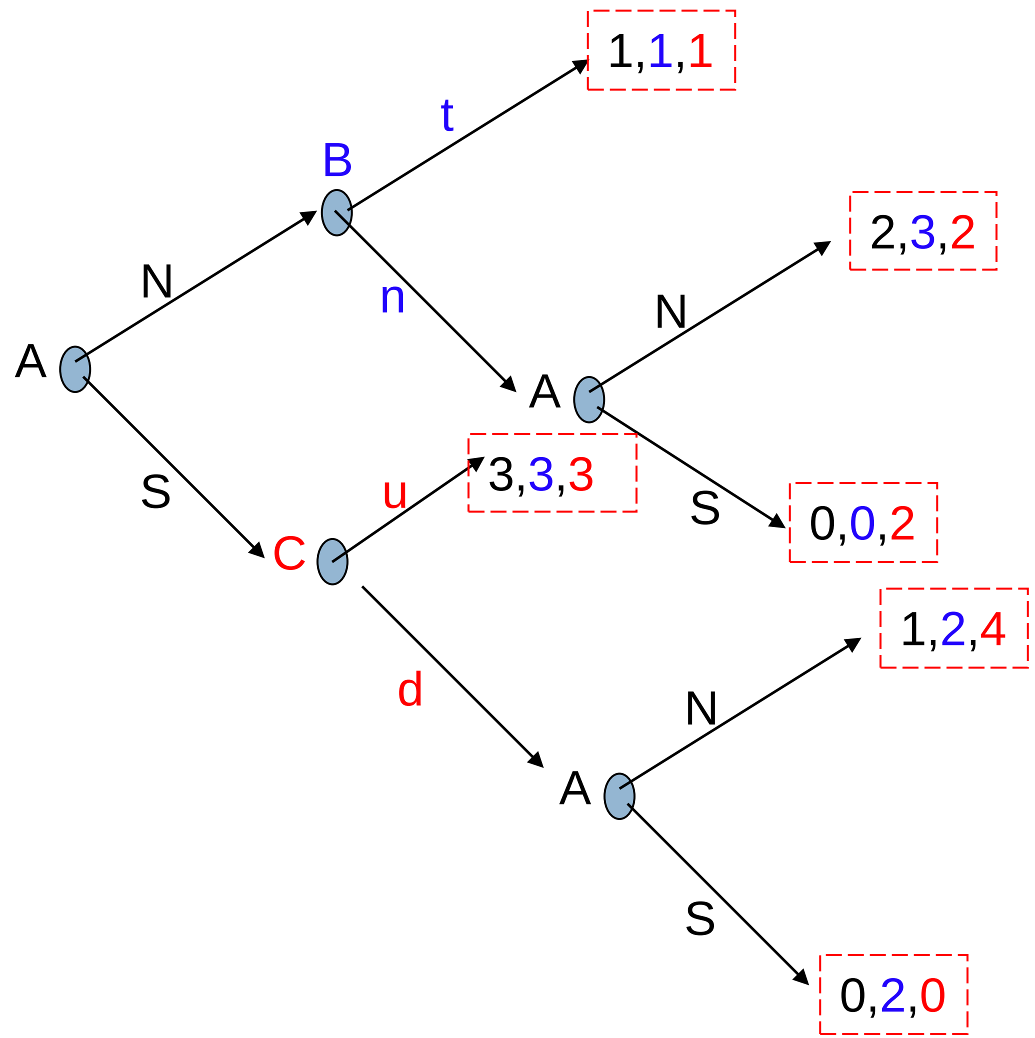
Backward Induction

Use Backward Induction

What is the Nash equilibrium?



Backward Induction



What is the Nash equilibrium?

Airbus-Boeing example

- Consider the rivalry between Airbus and Boeing to develop a new commercial jet aircraft.
  - Suppose Boeing is ahead in the development process, and Airbus is considering whether to enter the competition.
  - If Airbus stays out, it earns zero profit while Boeing enjoys a monopoly and earns a profits of $1 billion.
  - If Airbus decides to enter and develop the rival airplane, then Boeing has to decide whether to accommodate Airbus peacefully, or to wage a price war.
  - In the event of peaceful competition, each firm will make a profit of $0.3 billion. If there is a price war, each will lose $0.1 billion because the prices of airplanes will fall so low that neither firm will be able to recoup its development costs.
- Draw the tree (extensive form) of this game.
- Find the rollback equilibrium using backward induction.

Number Between 1 and 10

- The following game is a finite, sequential game:
  - Two players, A and B, take turns choosing a number between 1 and 10 (inclusive).
  - A goes first.
  - The cumulative total of all the numbers chosen is calculated as the game progresses.
  - The winner is the player whose choice of number takes the total to 100 or more.
  - Play the game.
- Does this game have First Mover Advantage or Second-Mover Advantage?
- HINT: Use backward induction to prove it

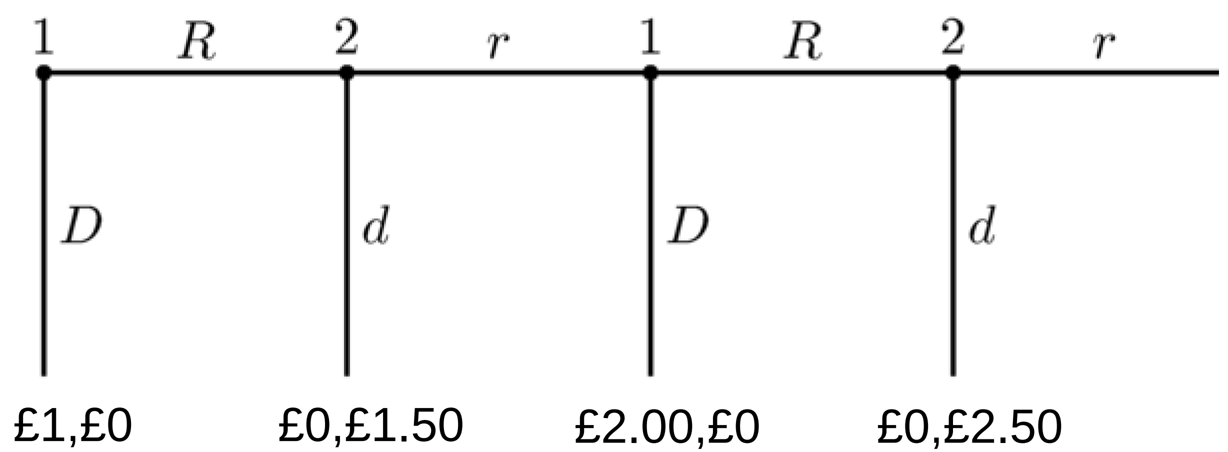
Number Between 1 and 10 - solution

- To win the game, you want to say the number that takes you to 100 or above:
  - So the only way I can do that is if my opponent gives me a number that is 90 or above
  - So to make sure HE gives ME 90 or above, I need to give him 89
  - He obviously knows this as this game contains FULL INFORMATION, and so is trying to give ME 89 at the same time (so that I can give him 90 or abvoe and he can win the game)
  - To avoid him giving me 89, I need to give him 78 – that way there is no way he can give me 89, and ensures I can give him the 89 instead
  - Using backward-induction (rollback analysis); we work towards the solution
  - 100→89→78→67→56→45→34→23→12→**1**
- **The person who picks 1 i.e. the FIRST MOVER can ALWAYS guarantee a win**

Centipede Game

How do the results change when you move to a finite game?

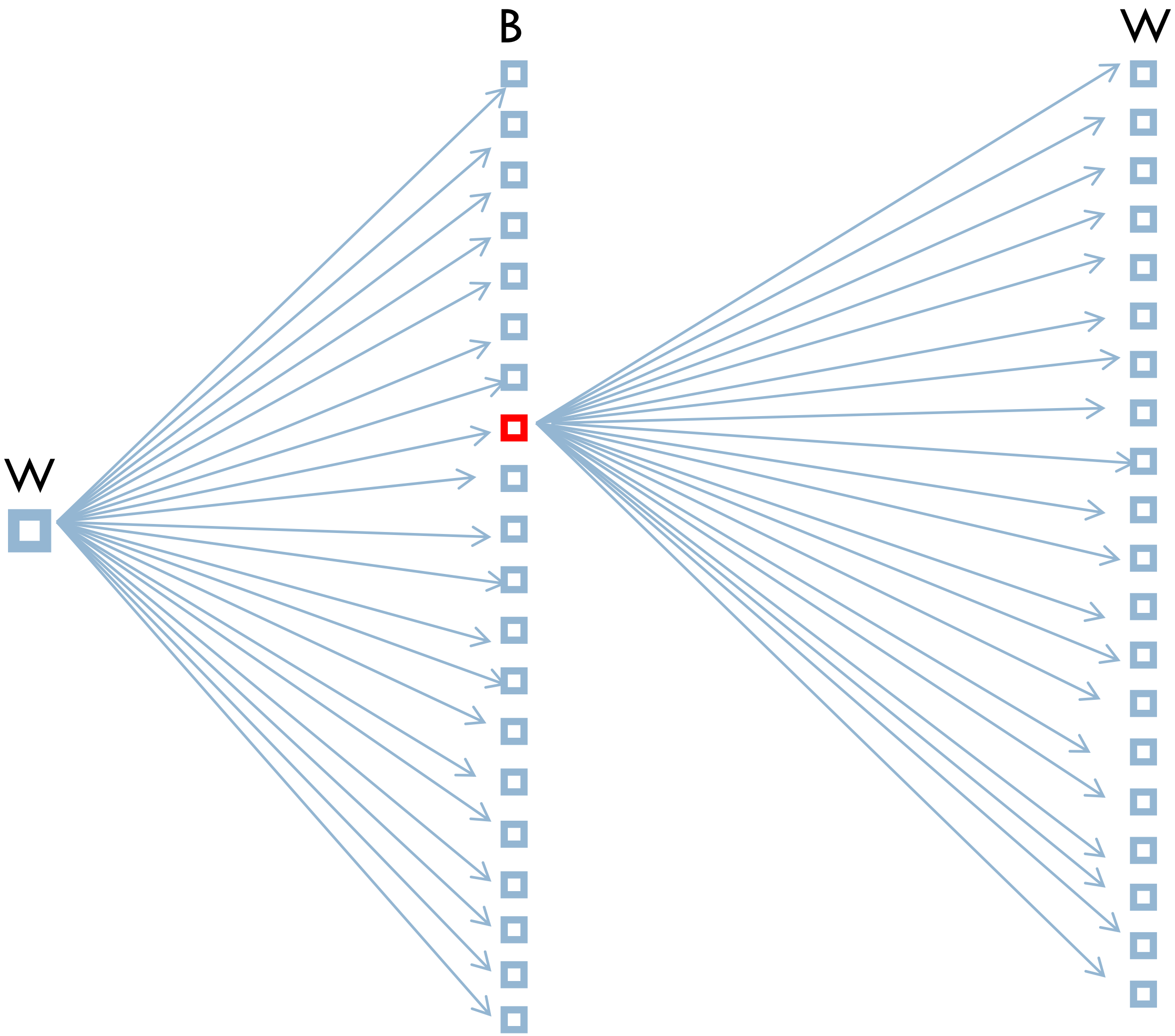
- The following game is a simultaneous game:
  - Assuming Extensive form game - in which two players take turns choosing either to accept some money, or pass it to the other player who has the option of accepting or rejecting it back to the original player.
  - E.g. I have £1, I can accept it, or pass it back to you when it becomes £1.50;
  - You can accept it or reject it back to me where it becomes £2.00;
  - I can accept it or reject it back to you where it becomes £2.50 etc etc.



Backward-induction

- We have shown very simply, that any game that is:
  - Sequential
  - Finite
  - Rationality
- Can be solved using backward-induction
  - This has stark implications for the game of CHESS
  - In theory, Chess can be solved
- The problem is that no computer is powerful enough to calculate the solution

Chess: Game Tree



- So just TWO moves into Chess, and you already have 400 possible outcomes
- The problem is that no computer is powerful enough to calculate the solution
- It is estimated that there are 10^120 total number of possible moves in Chess.
- The estimated number of atoms in the universe are 10^80. A supercomputer would take 3x10^103 years to check out all these moves (FYI: the Sun is predicted to die by then).

Prisoners' Dilemma

Pay-off matrix

Normal form

		B	
		Silent	Betray
A	Silent	6mths,6mths	10 years,0 years
	Betray	0 years,10 years	5 years, 5 years

REMEMBER: The first number always goes to the player on the LEFT, and the second number always goes to the player on the TOP. (e.g. 0,10 means A gets 0 years and B gets 10 years)

Rock-Paper-Scissors

- The following game is a simultaneous game:
  - Assuming you know this game, construct a normal form (table) payoff matrix for this game. What is the Nash equilibrium?
  - Rock beats Scissors.
  - Paper beats Rock.
  - Scissors beats Paper.
  - Play the game.
  - HINT: This is a zero-sum.

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

No Nash-equilibrium

Game of Chicken

- Refers to a game of “brinkmanship”
  - Traditional game of chicken refers to two cars driving to each other
  - The principle of the game is that while each player prefers not to yield to the other, the outcome where neither player yields is the worst possible one for both players.
  - This yields a situation where each player, in attempting to secure his best outcome, risks the worst.

Chicken

Normal form **Pay-off matrix**

		Dean	
		Swerve (Chicken)	Straight (Tough)
James	Swerve (Chicken)	0, 0	-1, 1
	Straight (Tough)	1, -1	-2, -2

A different pure strategy equilibrium is preferred by each player

Nuclear arms race

- Real-world example of brinkmanship is in the game between North Korea and the USA and the nuclear weapons issue

		N. Korea	
		Not build	Build
U. S.	Not Build	4, 4	1, 3
	Build	3, 1	2, 2

Rationality?

“Trembling Hand”

- Refers to a game of “brinkmanship”
  - Trembling hand perfect equilibrium is a refinement of Nash Equilibrium
  - A trembling hand perfect equilibrium is an equilibrium that takes the possibility of off-the-equilibrium play into account by assuming that the players, through a "slip of the hand" or tremble, may choose unintended strategies
  - So a player may have a dominant strategy, but may play a different strategy “with a tremble of the hand”
  - Allows for “irrational” play
  - The game represented in the following normal form matrix has two pure strategy Nash equilibria, namely <Up, Left> and <Down, Right>. However, only <U,L> is trembling-hand perfect.

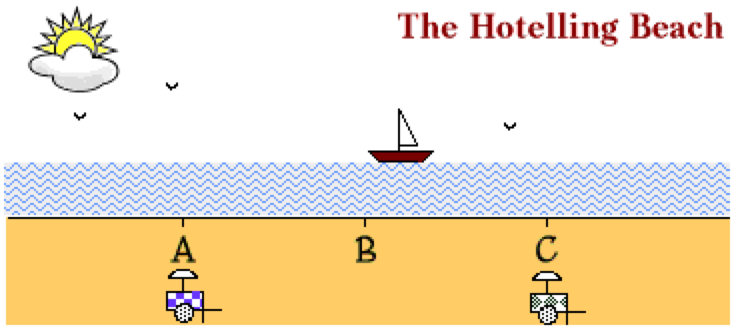
	Left	Right
Up	1, 1	2, 0
Down	0, 2	2, 2

Nash vs Bayesian Equilibria

- Applies to games of incomplete information
  - Use probabilistic analysis
  - Update their beliefs as per Bayes' Rule as the game progresses
  - A perfect Bayesian equilibrium is a strategy profile and a belief system such that the strategies are sequentially rational given the belief system and the belief system is consistent, wherever possible, given the strategy profile.

Hotelling's Model

Suppose that two ice-cream sellers, George and Henry, are trying to decide where to locate along a stretch of beach. Suppose further that there are 100 customers located at even intervals along this beach, and that a customer will buy only from the closest vendor. What is the Nash equilibrium?



Applications to the real world?

Guess 2/3 of the Average

Choose a number between 0 and 100.

A prize of 20 sweets will be split equally between all students whose number is closest to 2/3 of the average (mean) of the numbers chosen.

What do you choose?

Write down your answer.

What is the Nash equilibrium?

Game Theory – Keynes - Investments

John Maynard Keynes:

"...professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view.

It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest.

We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be."

(The General Theory, p.156, 1936.

Split the Sweets game

I offer you and your partner 5 sweets to split.

The first person offers a split to the second.

The second person has to accept the split.

FMA vs SMA

Play the game

I offer you and your partner 10 sweets to split.

The first person offers a split to the second.

The second person can accept OR reject the split, in which case both people get nothing.

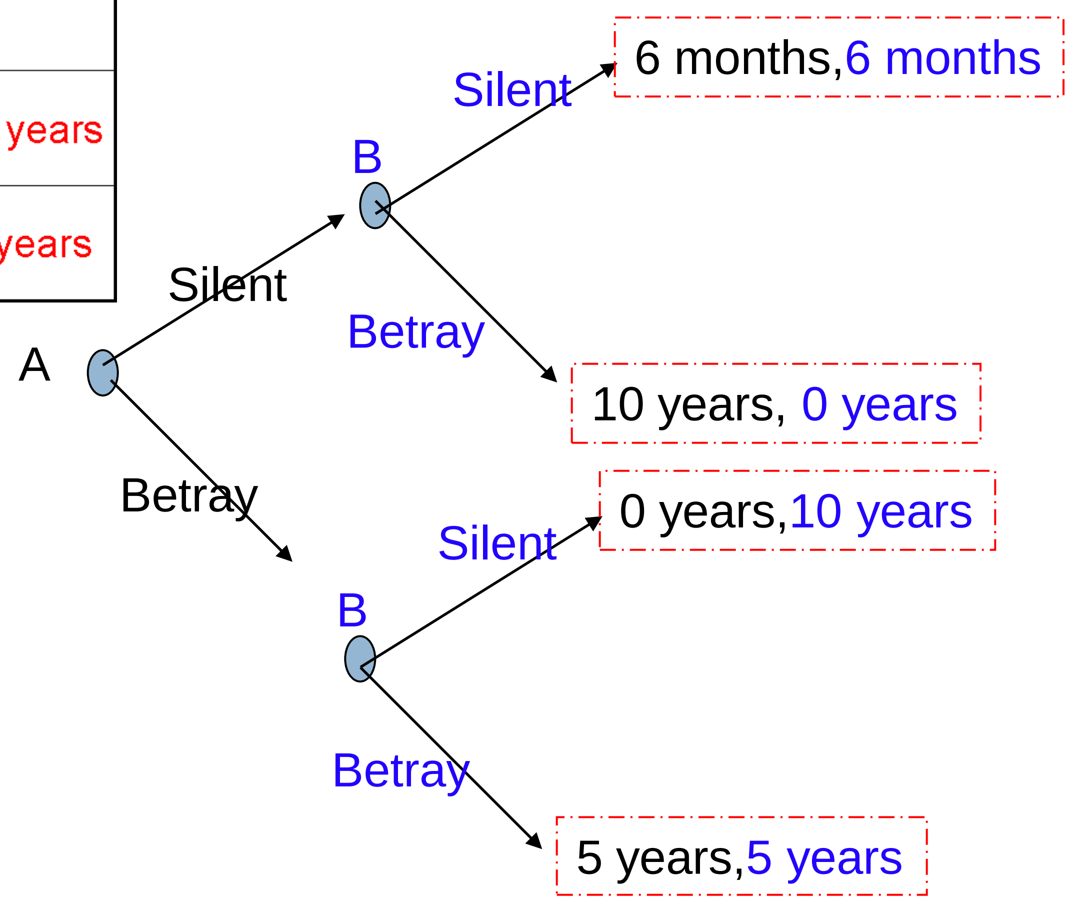
Repeated game vs One-shot game?

Play the game

P.D - Simultaneous-sequential game

- The following game is a simultaneous game:

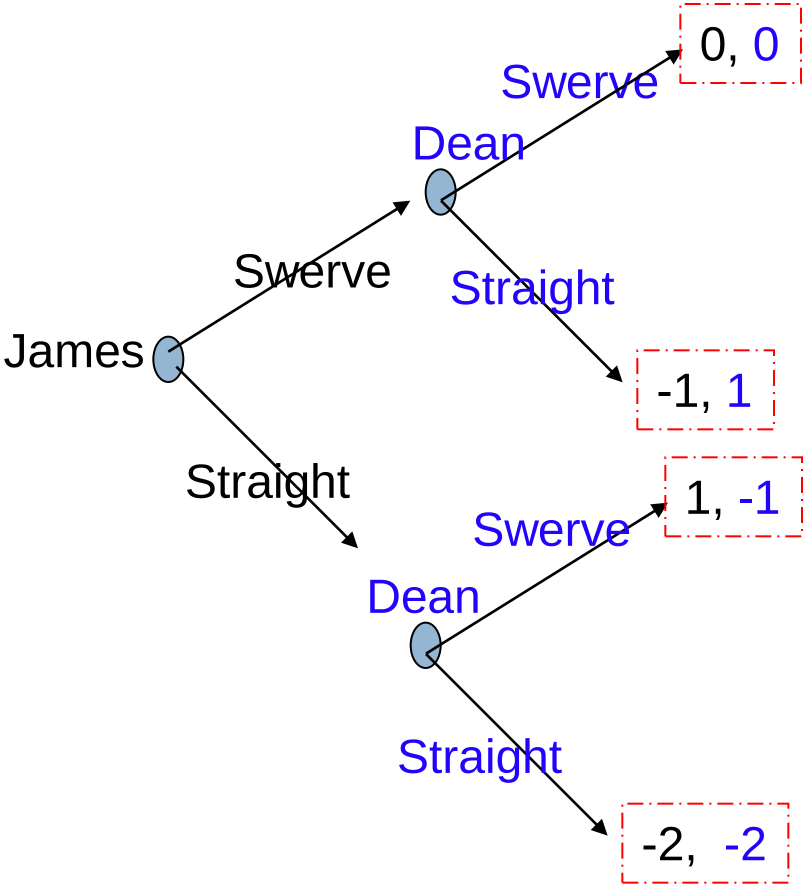
		B	
		Silent	Betray
A	Silent	6mths,6mths	10 years,0 years
	Betray	0 years,10 years	5 years, 5 years

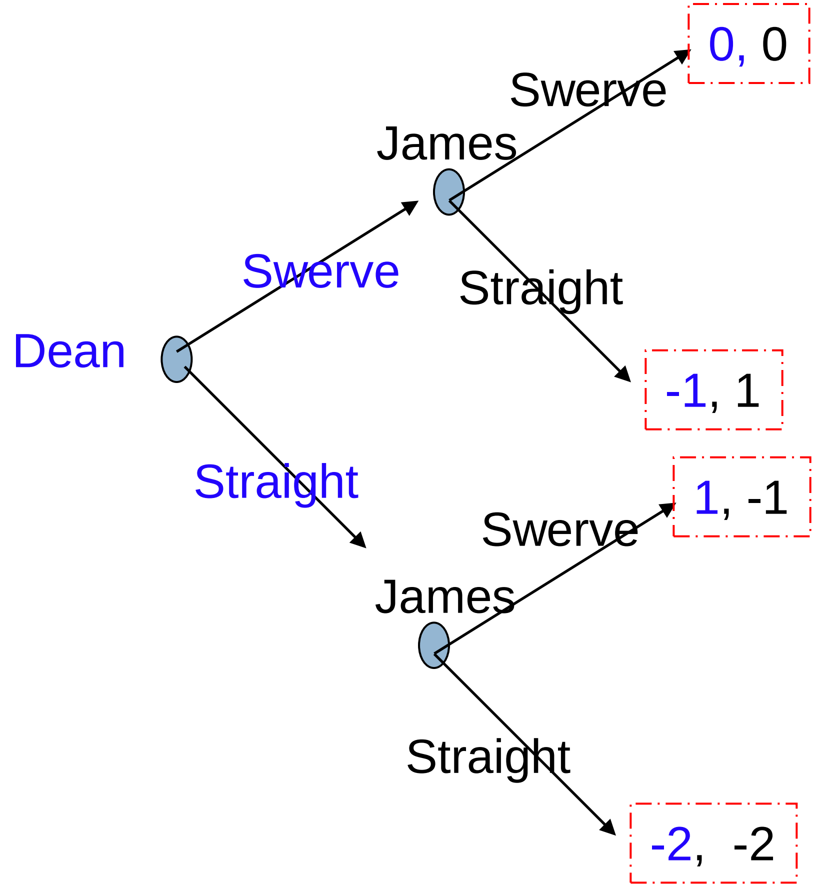


What's the Nash? Does it change when we move from simultaneous to sequential? Why?

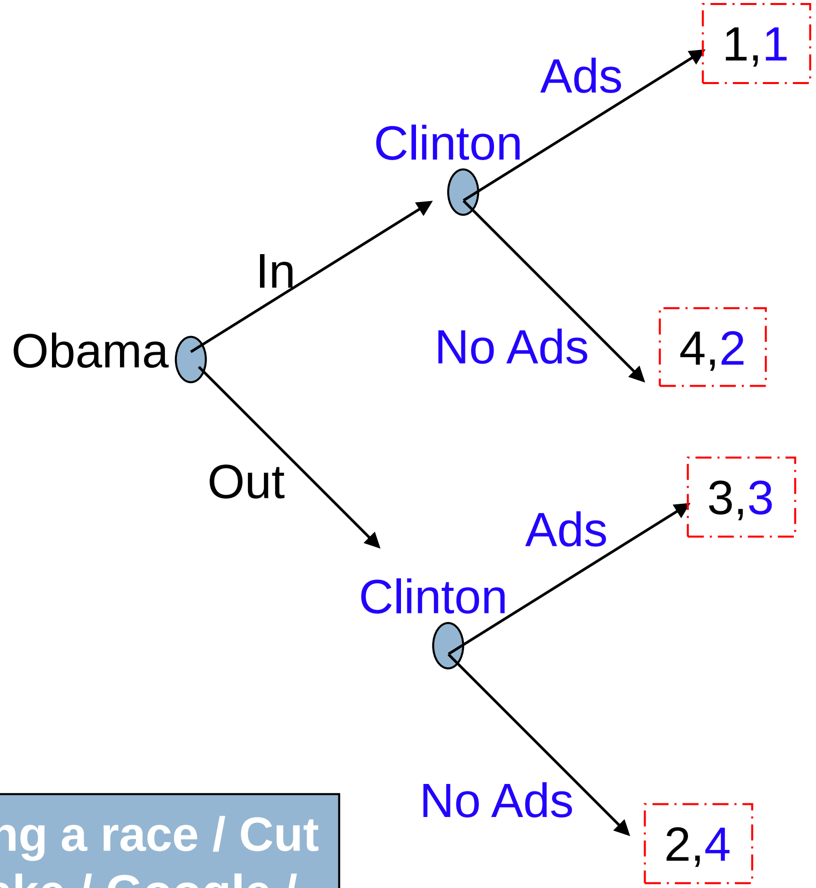
Simultaneous-sequential game: Chicken

What's the Nash in each of these cases? Is there First mover or second mover advantage?





FMA vs SMA



What is the Nash equilibrium?

Obama can look forward to Clinton's optimal responses and make a first move that is to his own advantage

FMA – 'set the stage' right

SMA – 'ability to react optimally'

Running a race / Cut the cake / Google / 3G / XBOX

Pizza Pies

- Consider a small college town with a population of dedicated pizza eaters but able to accommodate only two pizza shops, Mo's Pizza Pies and Francesca's Delicious Pizzas.
- Each seller has to choose a price for its pizza.
- To keep things simple we will assume that only 3 prices are available (H, M, L).
- If a high price is set, the sellers can achieve a profit margin of $12 per pie.
- The med price = $10 per pie.
- The low only $5 per pie.

- Each store has a loyal captive consumer base who will buy 3,000 pizzas every week, no matter what price is charged in either store.
- There is also a floating demand of 4,000 pies a week.
- The people who buy these pizzas are price-conscious and will go to the store with the lowest price. If both stores charges the same price, they will be split equally between the stores.

- If Francesca goes M and Mo goes H:
- Francesca has her 3000 + 4000 (at $10) = $70,000
- Mo only has his 3000 (at $12) = $36,000.

- Payoff matrix overleaf

Pizza Pies

		Mo's Pizza Pies		
		High	Med.	Low
Francesca's Deep Dish	High	60, 60	36, 70	36, 35
	Med.	70, 36	50, 50	30, 35
	Low	35, 36	35, 30	25, 25

Check for a Dominant Strategy (for either player)

Check for DominATED strategy (for either player)

Find the Nash equilibrium.

Remember: A dominated strategy is NEVER played. So you can eliminate it.

Battle of the Sexes

The Battle of the Sexes is a two-player coordination game used in game theory. Imagine a couple. The husband would most of all like to go to the football game. The wife would like to go to the opera. Both would prefer to go to the same place rather than different ones. If they cannot communicate, where should they go?

		Male	
		Opera	Football
Female	Opera	3, 2	0, 0
	Football	0, 0	2, 3

Find the Nash equilibrium (hint: may be more than 1!) ☺

But players prefer *different* equilibrium

Find all the Nash equilibria

		Column	
		Left	Right
Row	Up	1, -1	4, -4
	Down	2, -2	3, -3

What is the Nash equilibrium?

Check for dominant… or dominated strategies…

Find all the Nash equilibria

		Column		
		Left	Middle	Right
Row	Up	5, -5	3, -3	2, -2
	Straight	6, -6	4, -4	3, -3
	Down	1, -1	6, -6	0, 0

Check for dominant… or dominated strategies…

Find all the Nash equilibria

		Column	
		Left	Right
Row	Up	2, 4	1, 0
	Down	6, 5	4, 2

Check for dominant… or dominated strategies…

Find all the Nash equilibria

		Column		
		Left	Centre	Right
Row	Up	1, 2	2, 1	1, 0
	Straight	0, 5	1, 2	7, 4
	Down	-1, 1	3, 0	5, 2

Check for dominant… or dominated strategies…

Find all the Nash equilibria

		Column		
		Left	Centre	Right
Row	Up	4, 3	2, 7	0, 4
	Down	5, 5	5, -1	-4, -2

Check for dominant… or dominated strategies…

Letters game

Two players, Jack and Jill are put in separate rooms. Then each is told the rules of the game.

Each is to pick one of six letters, G, K, L, Q, R, or W.

If the two happen to choose the same letter, both get prizes as follows:

Letter:	G	K	L	Q	R	W
Jack's prize:	3	2	6	3	4	5
Jill's prize:	6	5	4	3	2	1

If they choose different letters, they get 0.
Everyone knows these payoffs.

Draw the table for this game. What are the Nash equilibria? Are any of them focal points?

Now play this game in pairs (one of you Jack, one of you Jill, and see what happens.

Find all the Nash equilibria

		Column	
		Left	Right
Row	Up	1, 1	0, 1
	Down	1, 0	1, 1

Check for dominant… or dominated strategies…

Find all the Nash equilibria

		Column	
		Left	Right
Row	Up	1, 1	0, 0
	Down	0, 0	0, 0

Battle of the Bismarck Sea (1943)

- Famous actual naval engagement in 1943, during WWII.
- The game in question – the battle of the Bismarck named for the body of water in the southwestern Pacific Ocean near Bismark Achipelago and Papua-New Guinea.
- In 1943, a Japanese admiral was ordered to transport troops and lead a supply convoy to New Guinea. The Japanese had a choice between a rainy northern route and a sunnier southern route.
- The U.S air forces knew that the convoy would sail and wanted to send bombers after it, but they did not know which route the Japanese would take.
- The Americans had to send reconnaissance aircraft to scout for the Japanese, but they had only enough planes to explore one route at a time.
- The sailing time was 3 days.
- If the Japanese convoy was on the route that the Americans explored first, the US could send its bombers straightaway; if not, a day of bombing was lost by the Americans.
- In addition, the poor weather on the northern route made it likely that visibility would be too limited for bombing on one day in 3.
- Thus the Americans could anticipate two days of active bombing if they explored the Northern route but discovered that the Japanese had gone South.
- If the Americans explored the Southern route first and found the Japanese there, they could get in 3 days of bombing; but if they found that the Japanese had gone north, they would get only 1 single day of bombing completed.
- Because the Japanese had to choose their route without knowing which direction the Americans would search first, and because the Americans had to choose the direction of their initial reconnaissance without knowing which way the Japanese had gone, this was a SIMULTANEOUS game.
- Hence payoff matrix overleaf.

Battle of the Bismarck Sea (1943)

		Japanese Navy	
		North	South
U.S.A.F.	North	2, -2	2, -2
	South	1, -1	3, -3

Check for dominant strategy for the U.S.A

Check for dominant strategy for the Japanese

Find the Nash equilibrium.

The Pirate Game

The pirate game is a simple mathematical game. It illustrates how, if assumptions conforming to a homo economicus model of human behaviour hold, outcomes may be surprising. It is a multi-player version of the ultimatum game.

The Game

- There are five rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.
- The Pirates have a strict order of seniority: A is superior to B, who is superior to C, who is superior to D, who is superior to E.
- The Pirate world's rules of distribution are thus: that the most senior pirate should propose a distribution of coins. The pirates should then vote on whether to accept this distribution; the proposer is able to vote, and has the casting vote in the event of a tie. If the proposed allocation is approved by vote, it happens. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again.
- Pirates base their decisions on three factors. First of all, each pirate wants to survive. Secondly, each pirate wants to maximize the amount of gold coins he receives. Thirdly, each pirate would prefer to throw another overboard, if all other results would otherwise be equal

The Pirate Game – The Result

The Result

•It might be expected intuitively that Pirate A will have to allocate little if any to himself for fear of being voted off so that there are fewer pirates to share between. However, this is as far from the theoretical result as is possible.

•This is apparent if we work backwards: if all except D and E have been thrown overboard, D proposes 100 for himself and 0 for E. He has the casting vote, and so this is the allocation.

•If there are three left (C, D and E) C knows that D will offer E 0 in the next round; therefore, C has to offer E 1 coin in this round to make E vote with him, and get his allocation through. Therefore, when only three are left the allocation is C:99, D:0, E:1.

•If B, C, D and E remain, B knows this when he makes his decision. To avoid being thrown overboard, he can simply offer 1 to D. Because he has the casting vote, the support only by D is sufficient. Thus he proposes B:99, C:0, D:1, E:0. One might consider proposing B:99, C:0, D:0, E:1, as E knows he won't get more, if any, if he throws B overboard. But, as each pirate is eager to throw each other overboard, E would prefer to kill B, to get the same amount of gold from C.

•Assuming A knows all these things, he can count on C and E's support for the following allocation, which is the final solution:

- •A: 98 coins; B: 0 coins; C: 1 coin; D: 0 coins; E: 1 coin

•Also, A:98, B:0, C:0, D:1, E:1 or other variants are not good enough, as D would rather throw A overboard to get the same amount of gold from B.

Game Theory – Keynes - Investments

- Investment game:
  - You can choose to invest in a stock market or not invest
- If you invest, it costs you $100
- If you choose not to invest it costs $0
- If you invest, you will get a return (net profit) of $50 IF at least 90% of your peers invest
- If less than 90% of people do not invest, you will lose your $100 (a -100 return)
- Let us play the game and see what happens

Applications to the real world?

Split the £ game

I offer you and your partner £10 to split.

You write down what share you want of it WITHOUT TELLING THE OTHER PERSON.

The aim, as always, is to maximise your OWN share.

You then reveal your answers.

If the shares equal less than £10, that is the split.

If the sum of the shares are more or equal to £10, you both get nothing (I get my £10 back).

Play the game

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