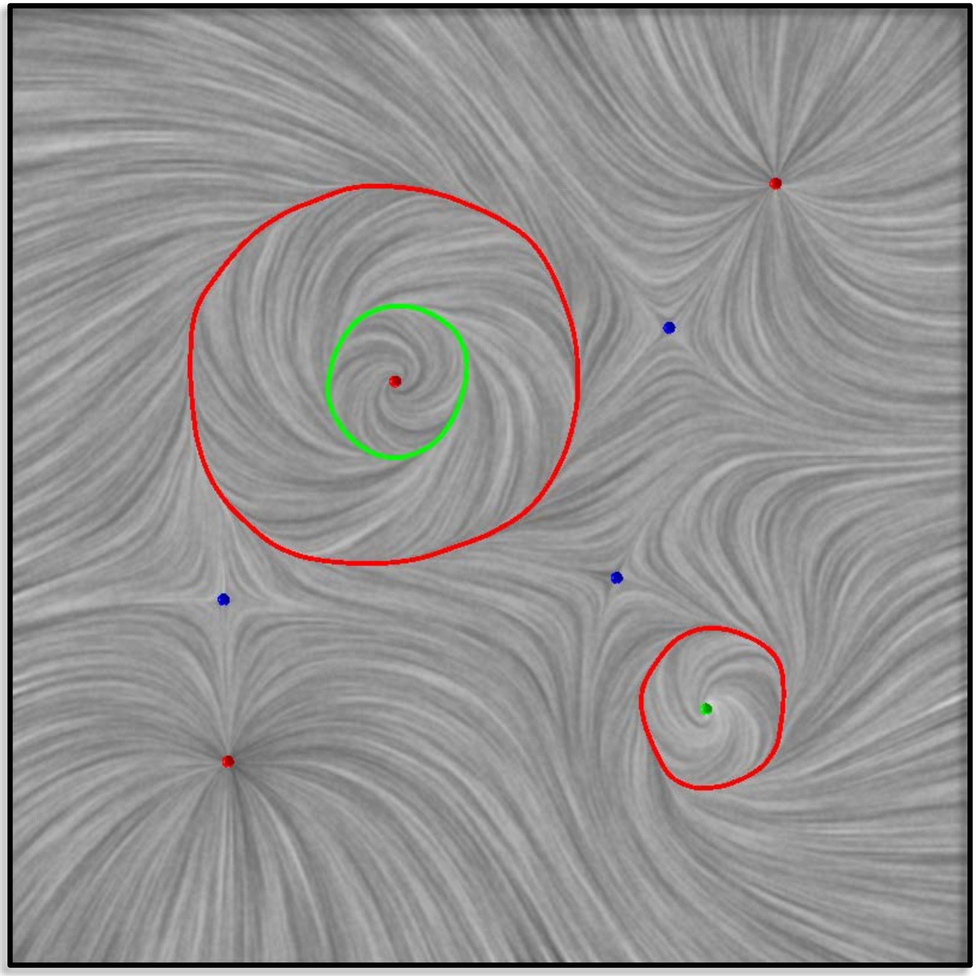
Vector Field Topology: Introduction

Let us focus on steady vector fields at this moment

Goal: understand what is vector field topology, what are fixed points and periodic orbits, how to extract and characterize fixed points and periodic orbits

What Are We Looking For From Flow Data?

- For steady flow



What Are We Looking For From Flow Data?

- For steady flow

Fixed points $V(x_0) = 0$

$\varphi(t, x_0) = x$ for all $t \in R$

- Sink
- Source
- Saddle

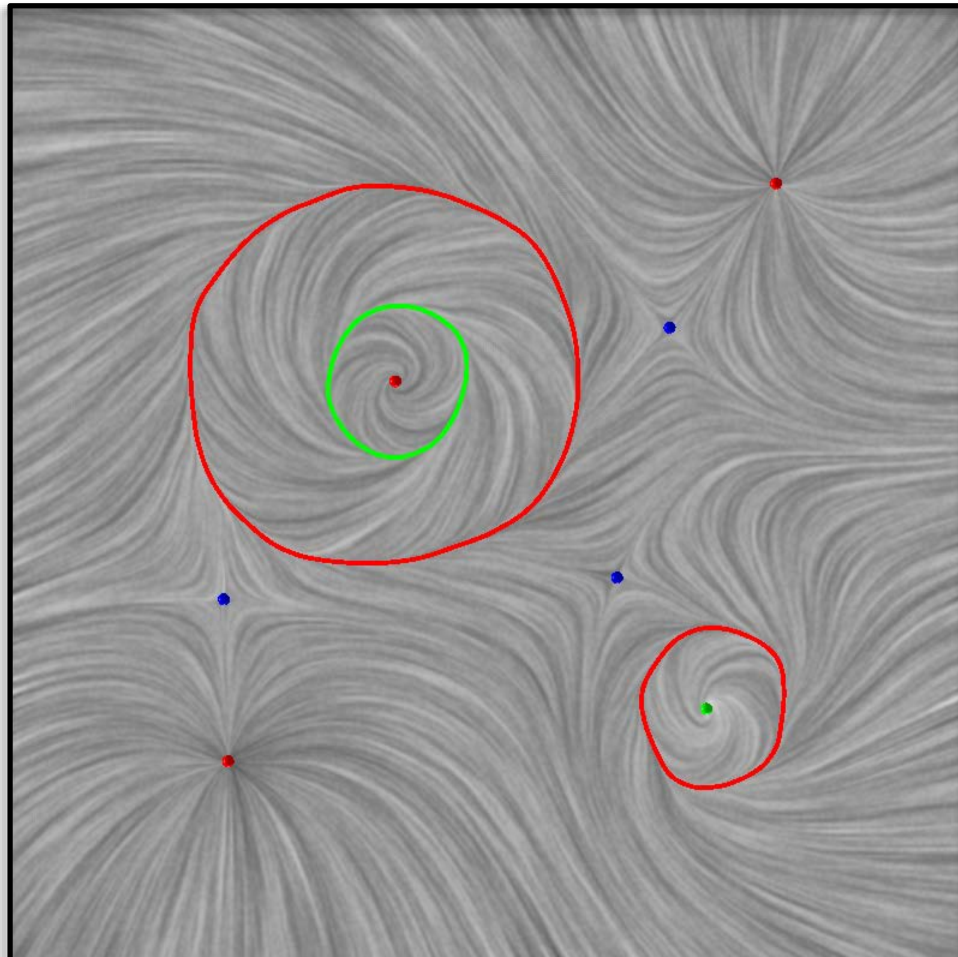
Periodic orbits

$\exists T_0 > 0$ such that $\varphi(T_0, x) = x$

 Attracting

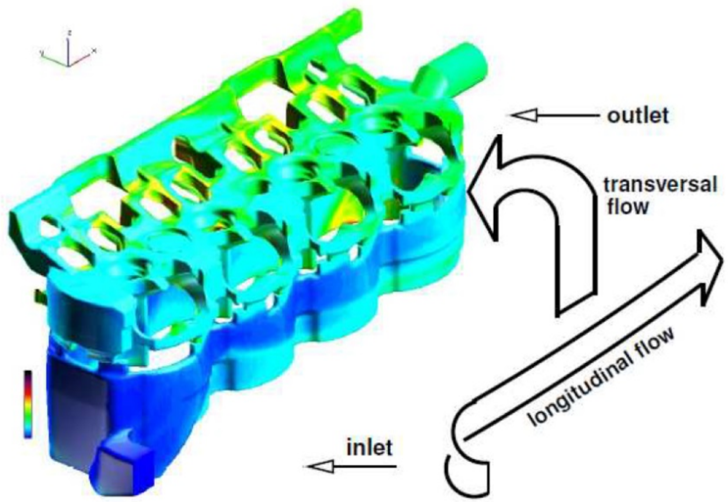
 Repelling

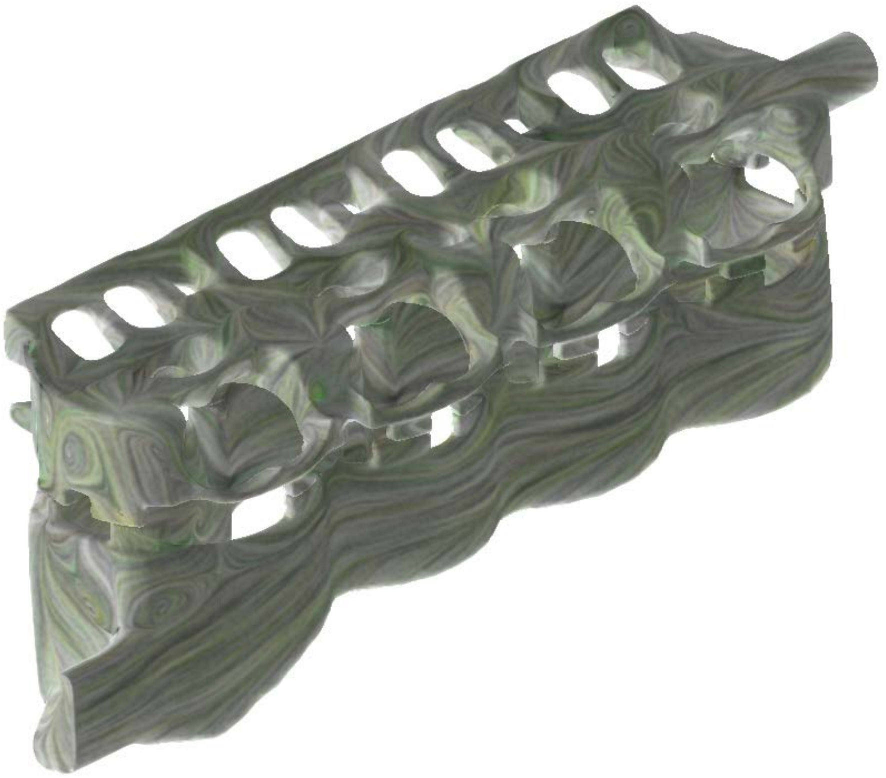
They are **flow recurrent** dynamics that trap flow particles forever



Example Application in Automatic Design

- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary

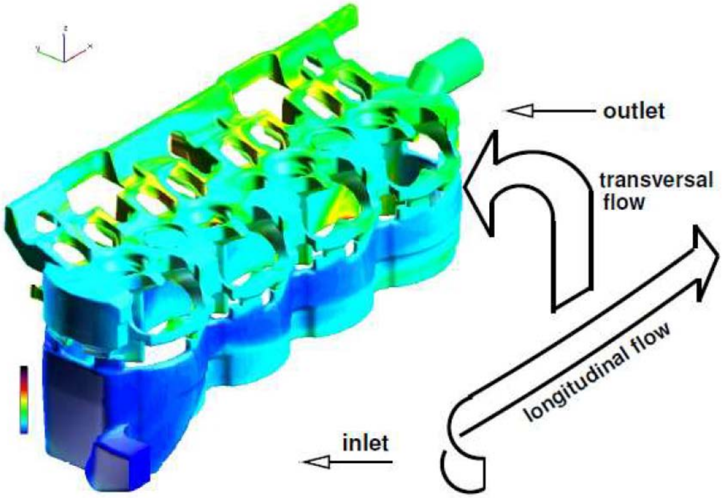


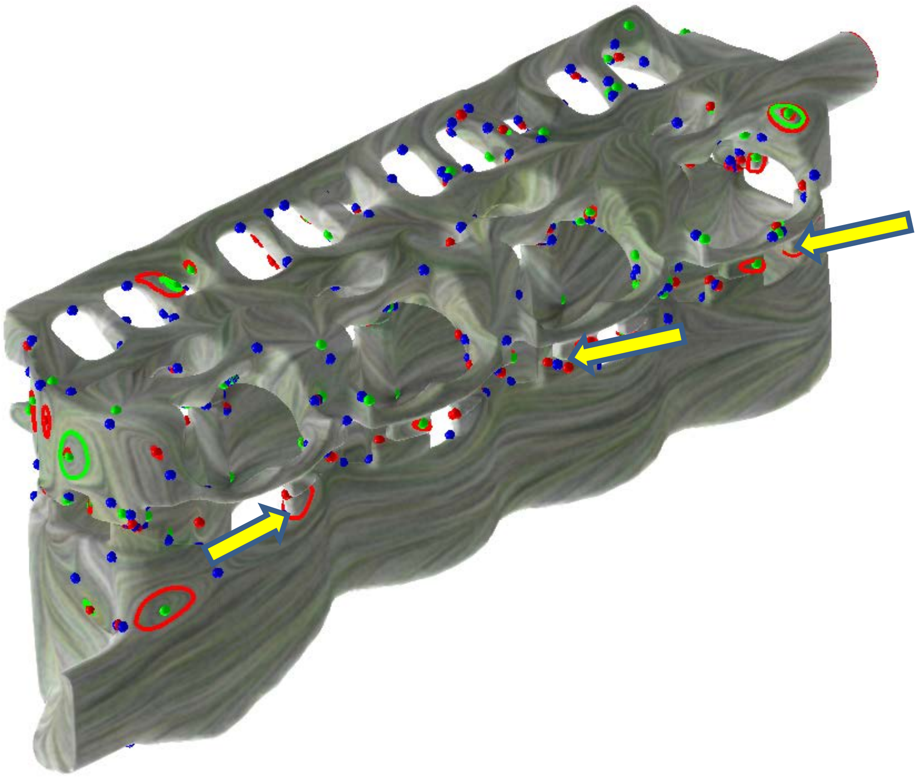


Where are the critical dynamics of interests?

Topology can help!

- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary

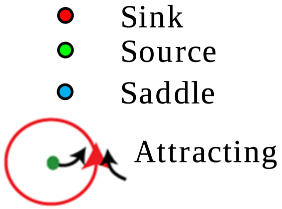




These critical dynamics are parts of vector field topology!

The connections of these (hyperbolic) flow recurrent features give rise to vector field topology!

- It condenses the whole flow information into its skeletal representation or structure, which is sparse.
- It provides a domain partitioning strategy which decomposes the flow domain into sub-regions. Within each sub-region, the flow behavior is homogeneous.
- It is one of those few rigorous descriptors of flow dynamics that are parameter free.
- It defines rigorous neighboring relations between features such that a hierarchy of the flow structure can be derived based on certain importance metric.
- *<u>This is what we need for large-scale data analysis in order to achieve multiscale/level-of-detail exploration!</u>*



Benefits

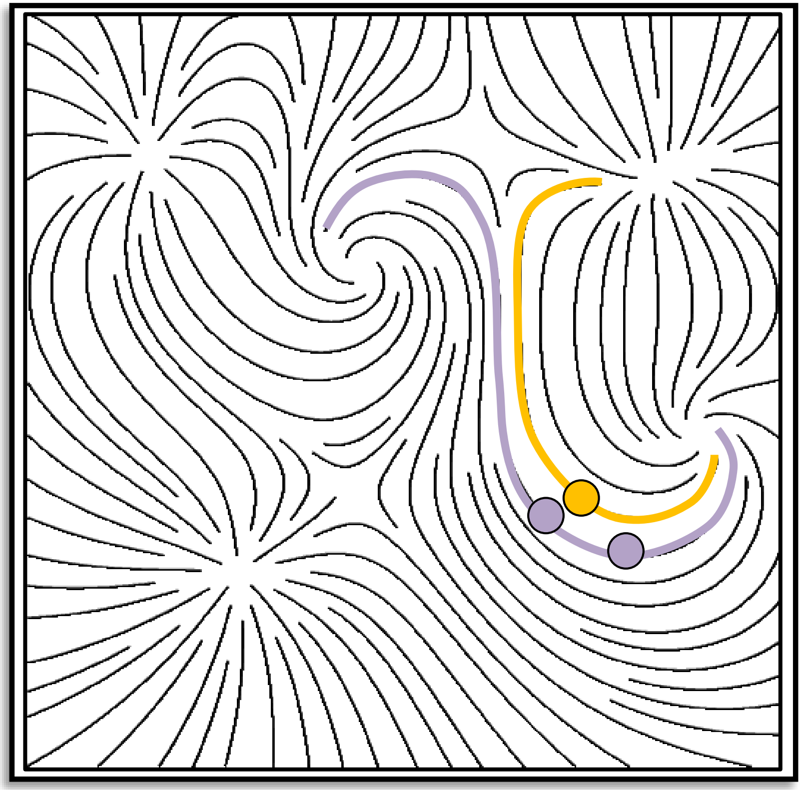
Some Theories

Vector Fields (Recall)

- A **vector field**
  - is a continuous vector-valued function $V(x)$ on a manifold X
  - can be expressed as a system of ODE $\dot{x} = V(x)$
  - introduces a **flow map** $\varphi : R \times X \rightarrow X$

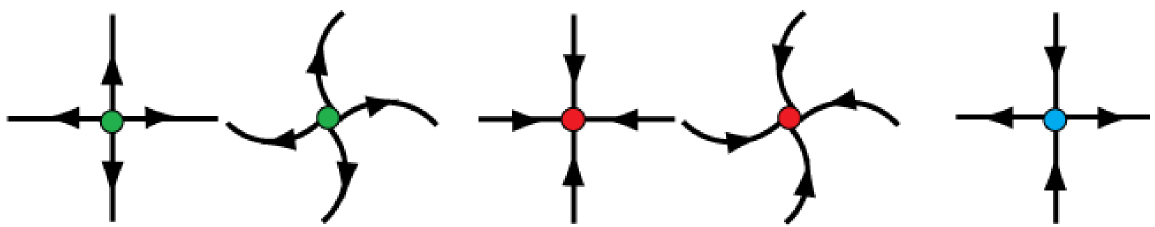
Recall—Trajectories

- A **trajectory** of $x \in X$ is $\cup_{t \in R} \varphi(t, x)$
- Given an initial condition, there is a unique solution
  $\mathbf{x}(t) = \mathbf{x}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{x}(u))\, \mathrm{d}u$
  $\varphi(t_0) = \mathbf{x}_0$
- Uniqueness
- Under time-independent setting a trajectory is also called **streamline**.



Fixed Points and Periodic Orbits

- A point $x \in X$ is a **fixed point** if $\varphi(t, x) = x$ for all $t \in \boldsymbol{R}$. This means that $\mathrm{V}(x)=0$.



(a) Source (b) Sink (c) Saddle

Fixed Points and Periodic Orbits

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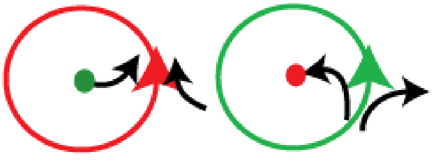
- x is a periodic point if there exist a $T > 0$ such that $\varphi(T, x) = x$. The trajectory of a periodic point is called a **periodic orbit**.



(d) Periodic orbits

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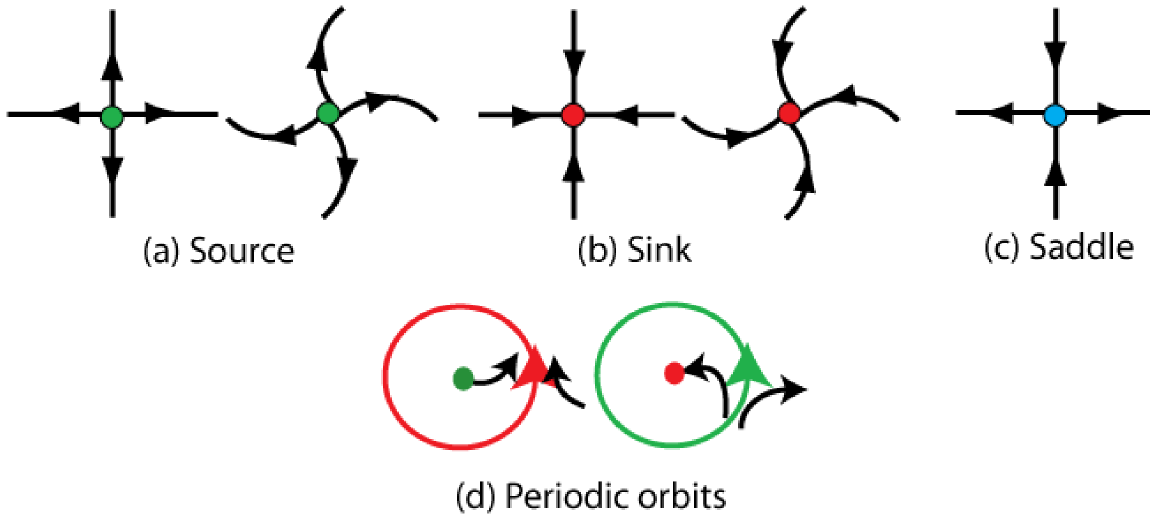
- *Both of them are **special trajectories***!



Limit Sets

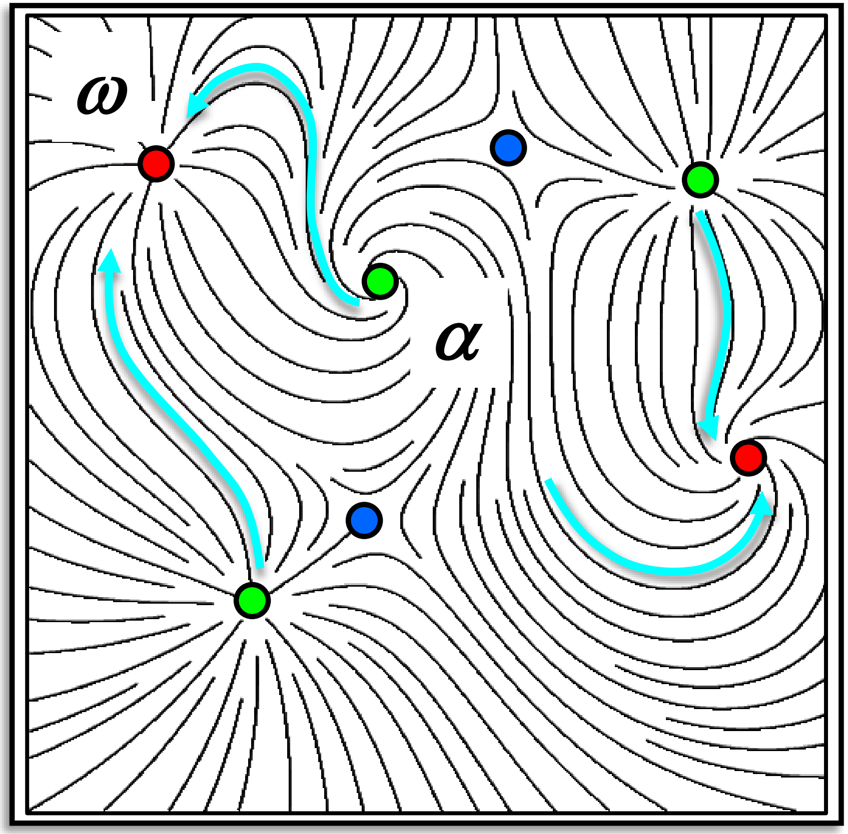
- **Limit sets** reveal the (**infinitely**) long-term behaviors of vector fields, correspond to flow recurrence.
- Two types of **limit sets** :

$$\alpha(\mathbf{x}) = \cap_{t<0}\, cl(\varphi((-\infty, t), \mathbf{x}))$$

point (or curve) reached after ***backward*** integration by streamline seeded at $\mathbf{x}$

$$\omega(\mathbf{x}) = \cap_{t>0}\, cl(\varphi((t, \infty), \mathbf{x}))$$

point (or curve) reached after ***forward*** integration by streamline seeded at $\mathbf{x}$



Invariant Sets

- An **invariant set** $S \subset X$ satisfies $\varphi(R,S)=S$
  - A trajectory is an invariant set
  - Fixed points and periodic orbits are *compact* and *disjoint* invariant sets

The characterization of different flow behaviors can be done by characterizing different trajectories, i.e. invariant sets!

In Practice

Fixed Point Extraction and Classification (Overview)

- Assume **piecewise linear** vector field. We adopt ***cell-wise*** analysis

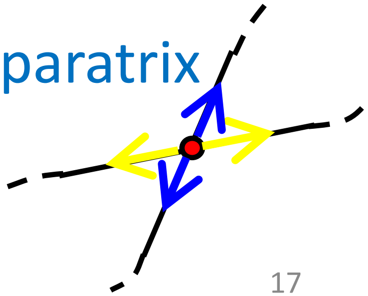
- Extraction $\vec{\mathbf{v}}(x_0, y_0) = \vec{0}$
  - Solve linear / quadratic equation to determine position of fixed point in cell

Fixed Point Extraction and Classification (Overview)

- Assume **piecewise linear** vector field. We adopt ***cell-wise*** analysis

- Extraction $\vec{\mathbf{v}}(x_0, y_0) = \vec{0}$
  - Solve linear / quadratic equation to determine position of fixed point in cell

- Classification
  - Compute **Jacobian** at that position
  - Compute **eigenvalues for classification**
  - If type is saddle, compute eigenvectors for separatrix computation

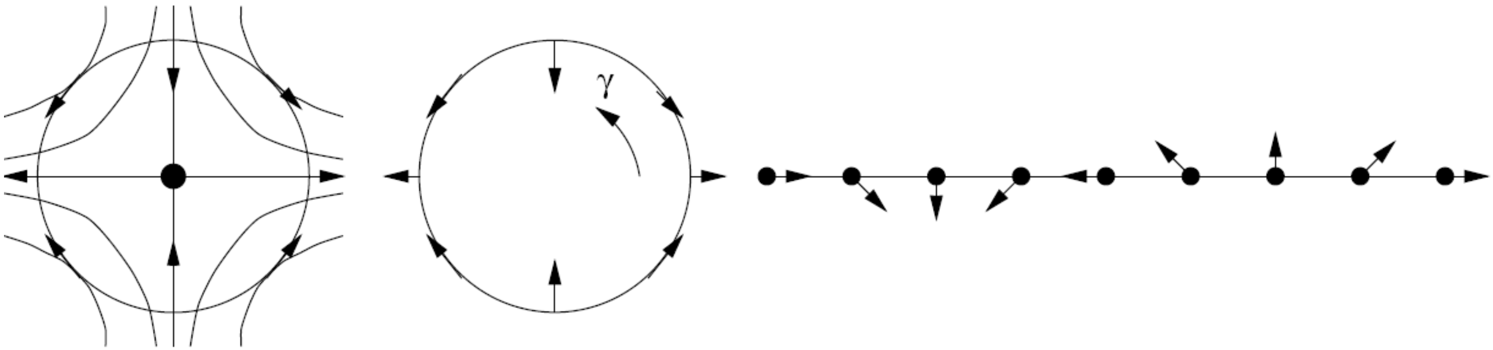


Solving the linear system for each cell is too expensive and wasting, as most cells will NOT contain fixed points.
But you do not know that until you solve for their linear systems.

Do we have better way to quickly identify cells that contain fixed points?

Poincaré Index

- **Poincarè index** I(Γ, V) of a simple closed curve Γ in the plane relative to a continuous vector field is the number of the positive field rotations while traveling along Γ in positive direction.



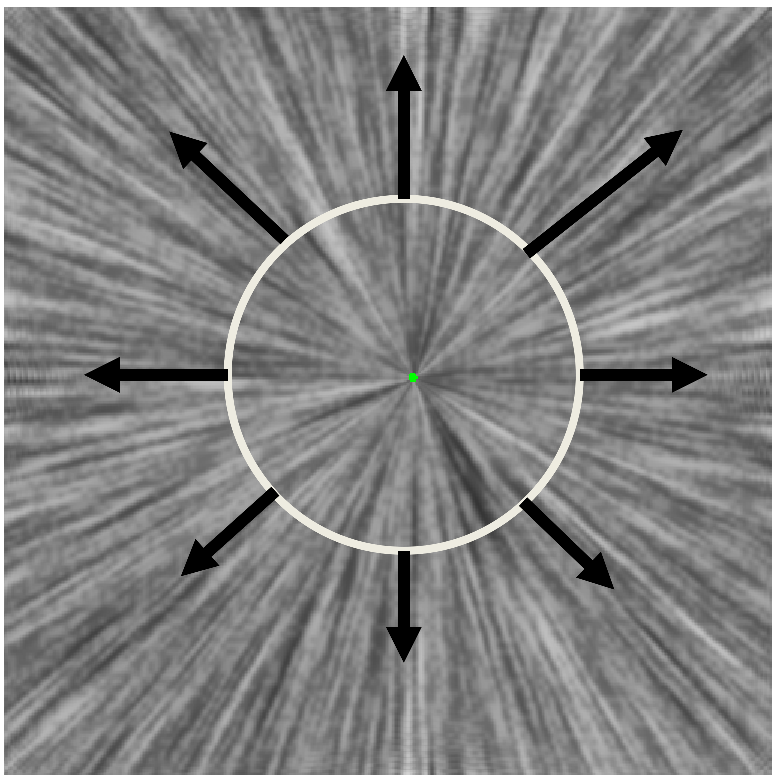
[Tricoche Thesis2002]

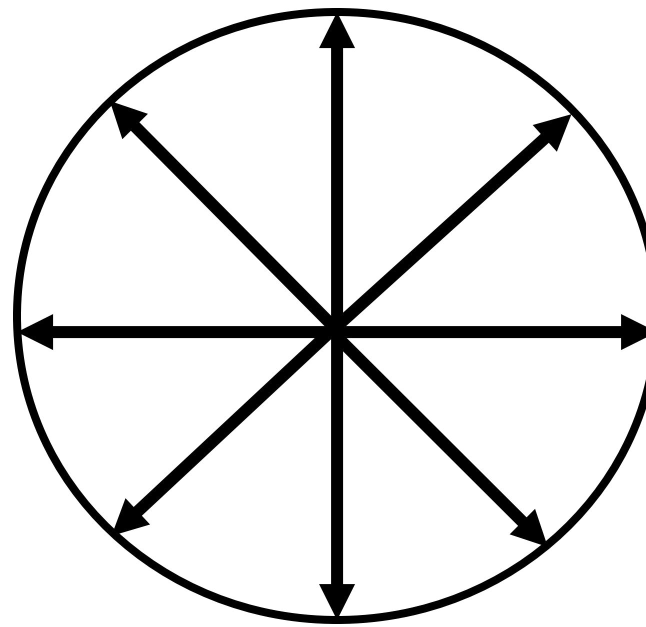
- By continuity, always an integer

- *The index of a closed curve around multiple fixed points will be the sum of the indices of the fixed points*

Poincaré Index

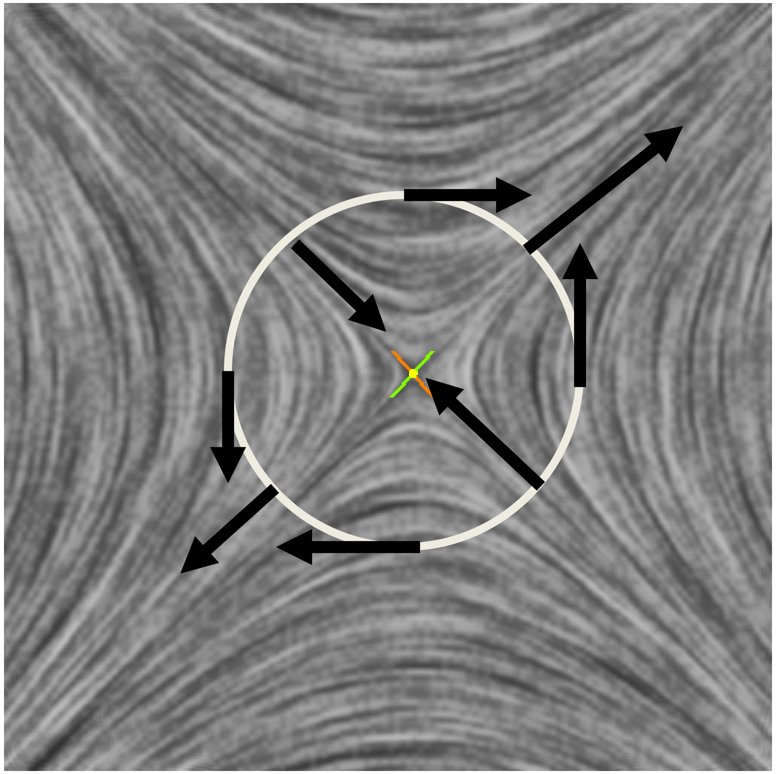
- Poincaré index: sources

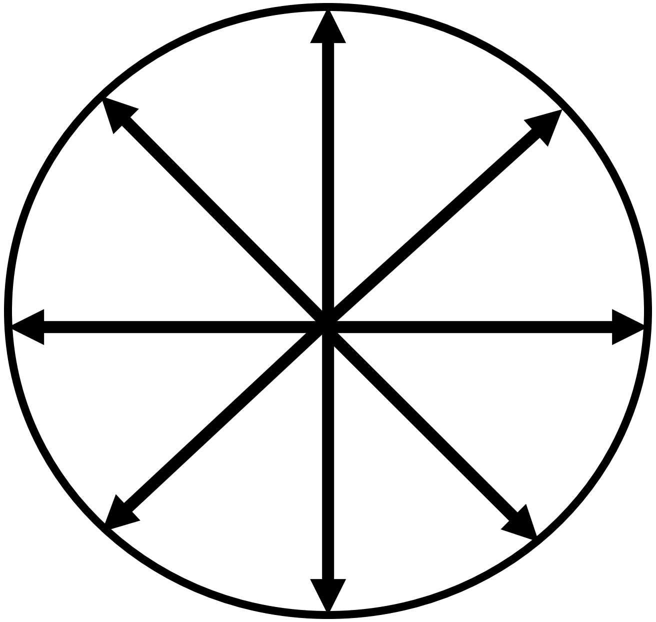




Poincaré Index

- Poincaré index: saddles

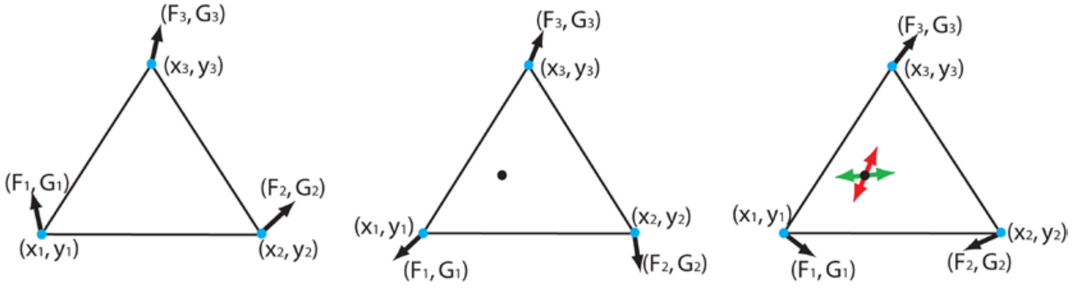




Important Poincaré Indices

- Consider an **isolated** fixed point x_0, there is a neighborhood N enclosing x_0 such that there are no other fixed points in N or on the boundary curve ∂N
  - if $\mathbf{I}(\partial N, V) = 1$, x_0 is either a source, a sink, or a center;
  - if $\mathbf{I}(\partial N, V) = -1$, x_0 is a saddle.

- The Poincarè index of a fixed point free region is 0



What are the Poincaré indices for these three regions?

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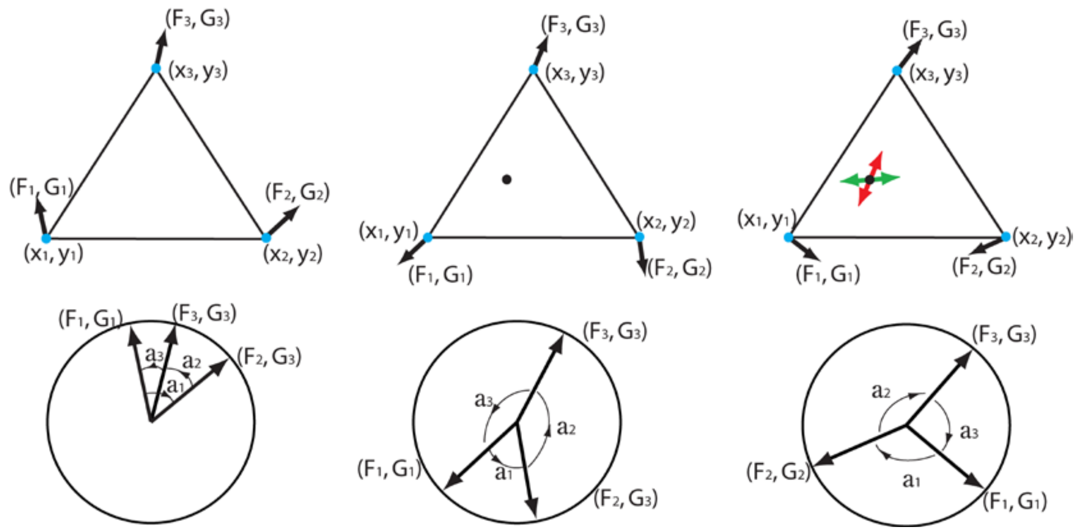
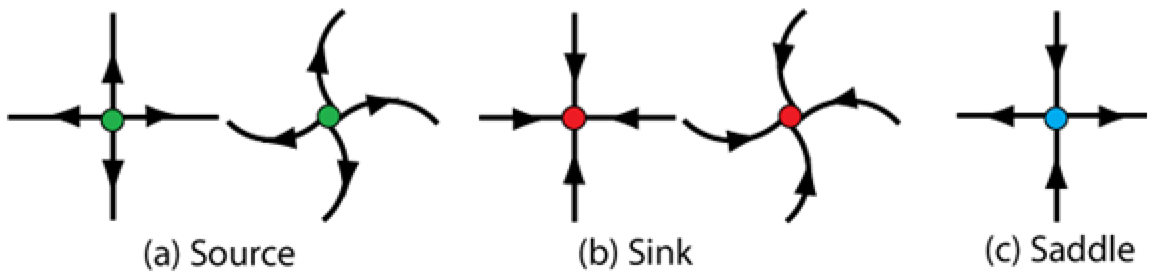


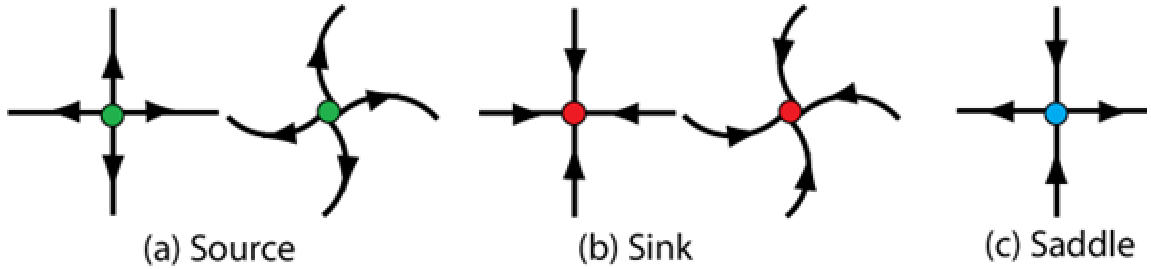
Fixed Point Classification



We specifically consider first-order fixed points -> Jacobian is not degenerate

$\det(J) = ae - bd \neq 0$ -> Jacobian matrix is full rank

Fixed Point Classification



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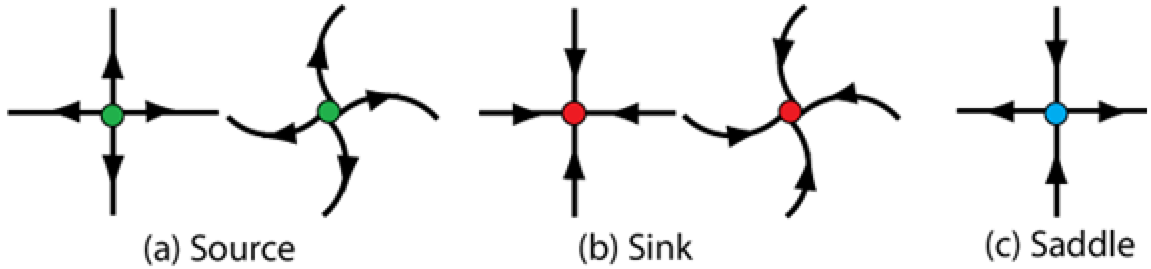
The eigenvalues of the Jacobian matrix $\lambda J = \lambda \boldsymbol{x}$ are

$$\lambda = Re_{1,2} + iIm_{1,2}$$

Fixed Point Classification



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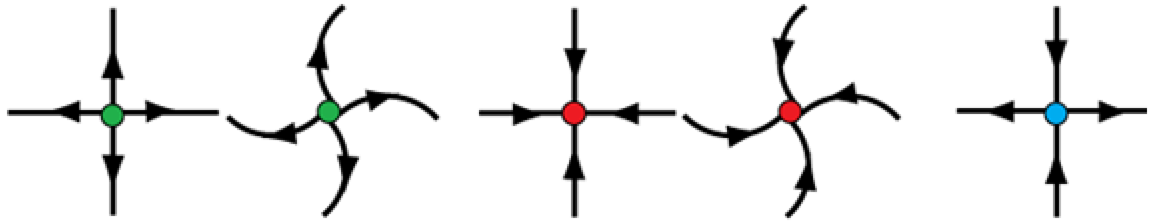
If both $Re_{1,2} > 0$, the fixed point repels flow locally.
If both $Re_{1,2} < 0$, the fixed point attracts flow locally.
If $Re_1 Re_2 < 0$, it does both and is a saddle

Fixed Point Classification



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If either $Re_{1,2} \neq 0$, the fixed point is called hyperbolic and stable.
If both $\boldsymbol{Re_{1,2} = 0}$ and $\boldsymbol{Im_{1,2} \neq 0}$, the fixed point is non-hyperbolic and unstable.

Fixed Point Classification

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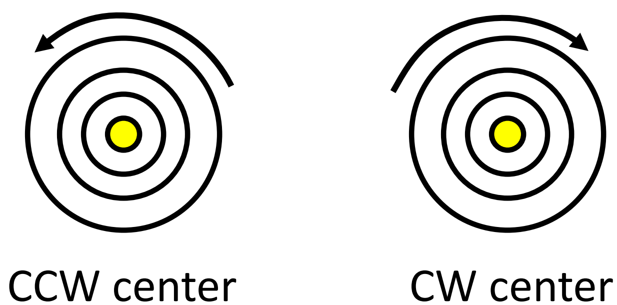

centers!

Can streamlines reach centers?

Normal Forms of Jacobians at Fixed Points

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Normal Forms of Jacobians at Fixed Points

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Source

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Sink

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Saddle

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Saddle

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

CCW center

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

CW center

Periodic Orbits

- Curve-type (1D) limit set
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Periodic Orbits

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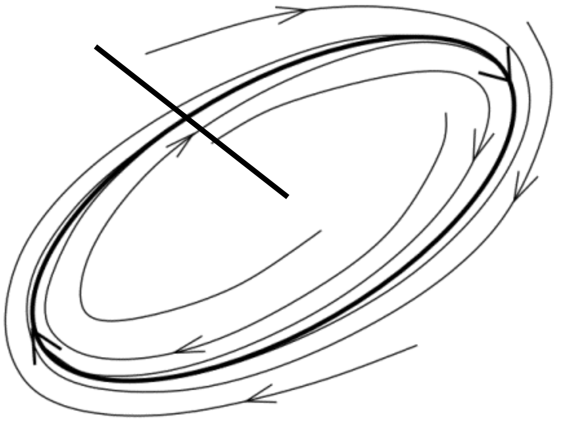
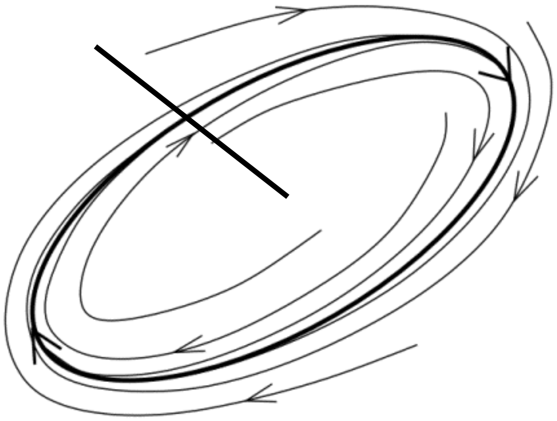
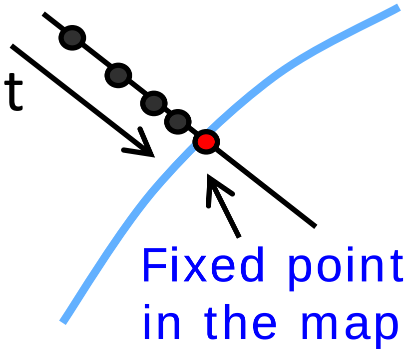
- **Poincaré map**:
  - Defined over cross section



Periodic Orbits

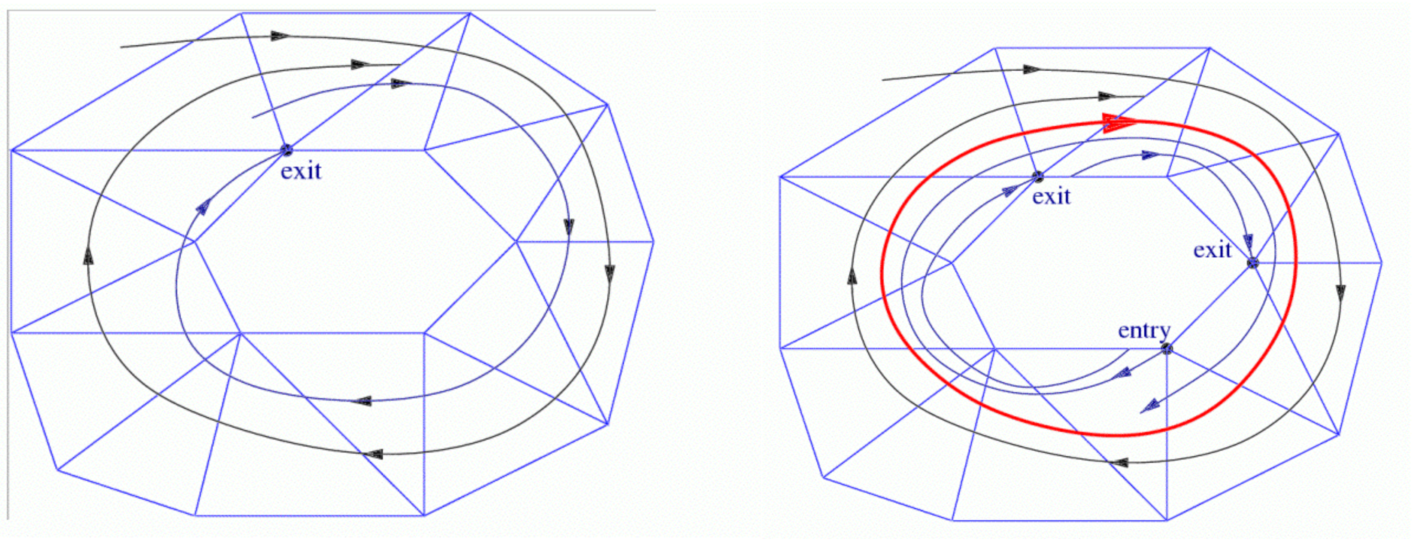


- Curve-type (1D) limit set
- Attracting / repelling behavior
- **Poincaré map**:
  - Defined over cross section
  - Map each position to next intersection with cross section along flow
  - Discrete map
  - The periodic orbit intersects at fixed point
  - Hyperbolic



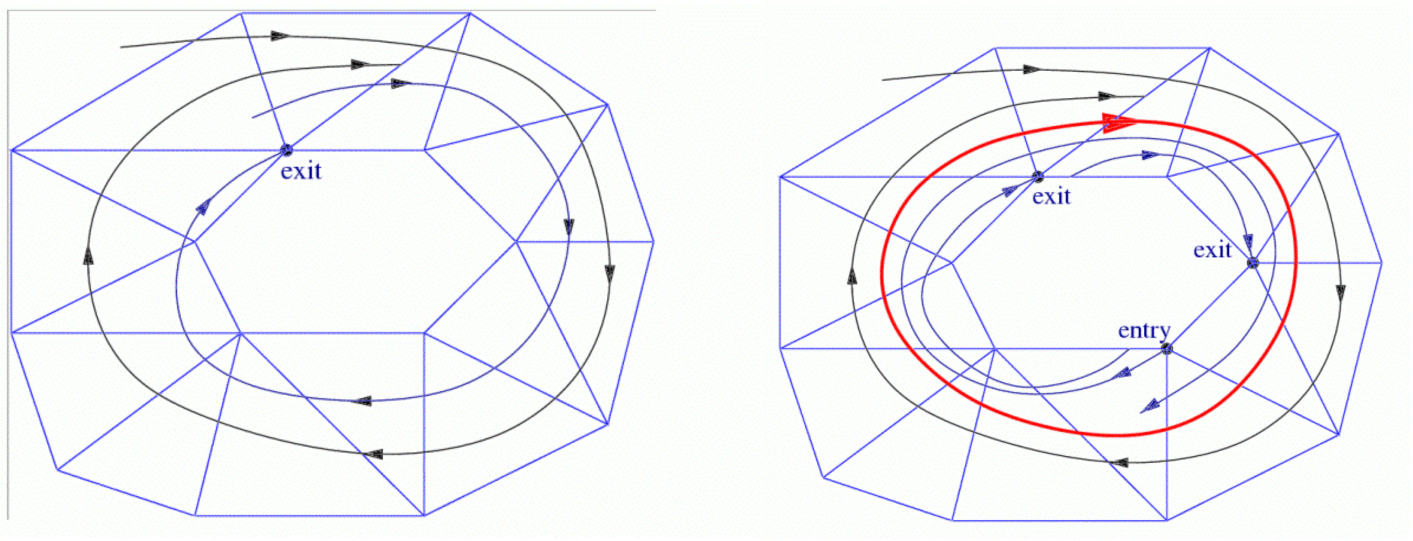
Periodic Orbit Extraction

- Poincaré-Bendixson theorem:
  – If **a region contains a limit set and no fixed point**, it contains a closed orbit



Periodic Orbit Extraction

- Detect closed cell cycle
- Check for flow exit along boundary
- Find exact position with Poincaré map(fixed point)



Some More Theories

Periodic Orbit Classification

Poincaré index for a periodic orbit is *0!*

Periodic Orbit Classification

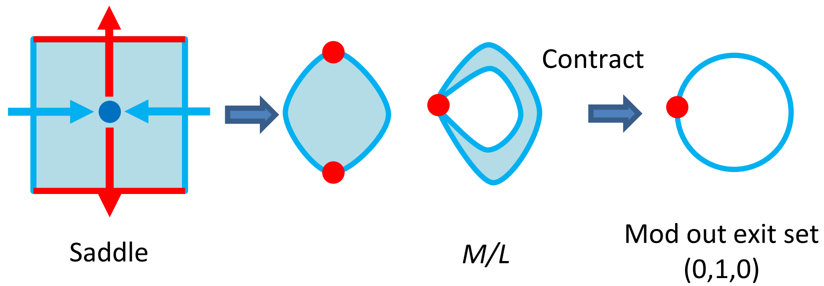
Poincaré index for a periodic orbit is *0!*

To address that, people introduce **the Conley index!**

$(\beta_0, \beta_1, \beta_2)$

If $\beta_0 = 1$, attracting

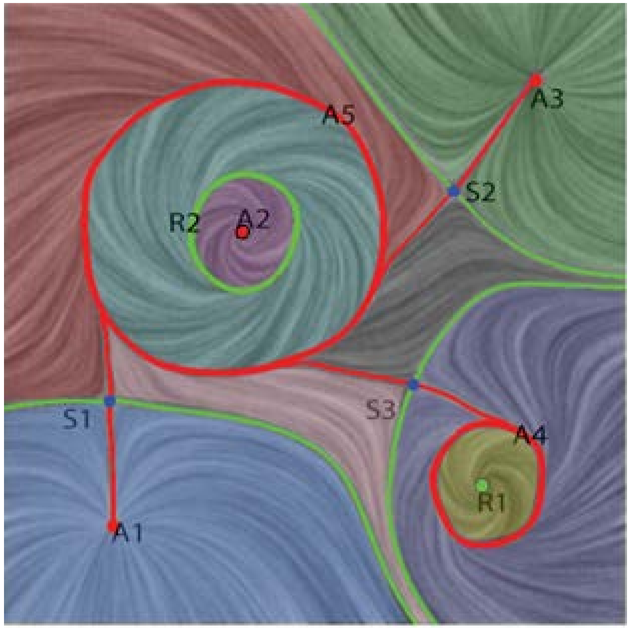
If $\beta_2 = 1$, repelling



Vector Field Topology

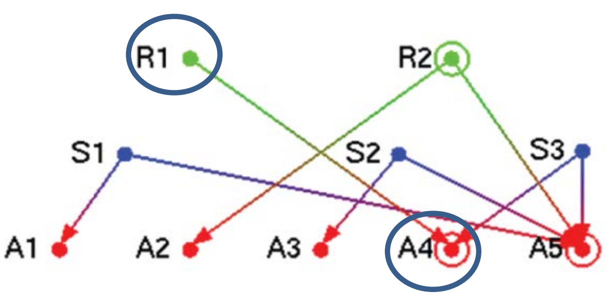
- Vector field topology provides qualitative (structural) information of the underlying dynamics

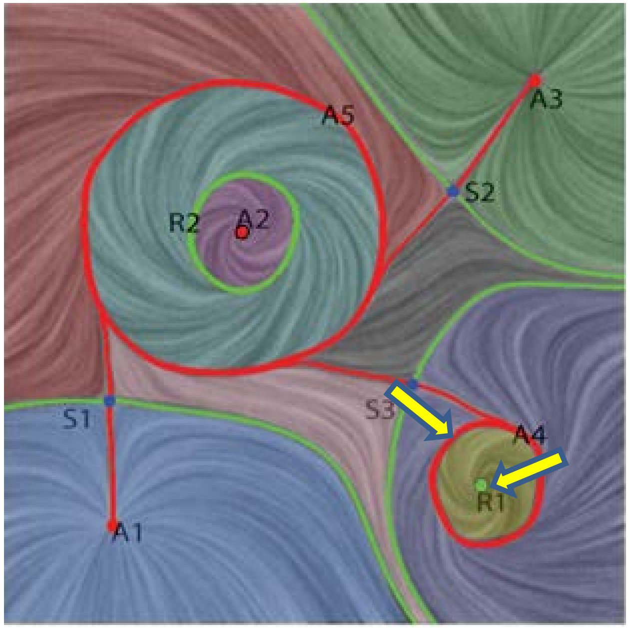
- It usually consists of certain critical features and their connectivity, which can be expressed as a graph, e.g. vector field skeleton [Helman and Hesselink 1989]
  - Fixed points (hyperbolic)
  - Periodic orbits (hyperbolic)
  - Separatrices



Topological Graph

- Three layers based on the Conley index
  - If $\beta_0 = 1$, (A)ttractors: sinks, attracting periodic orbits
  - If $\beta_2 = 1$, (R)epellers: sources, repelling periodic orbits
  - Otherwise, (S)addles

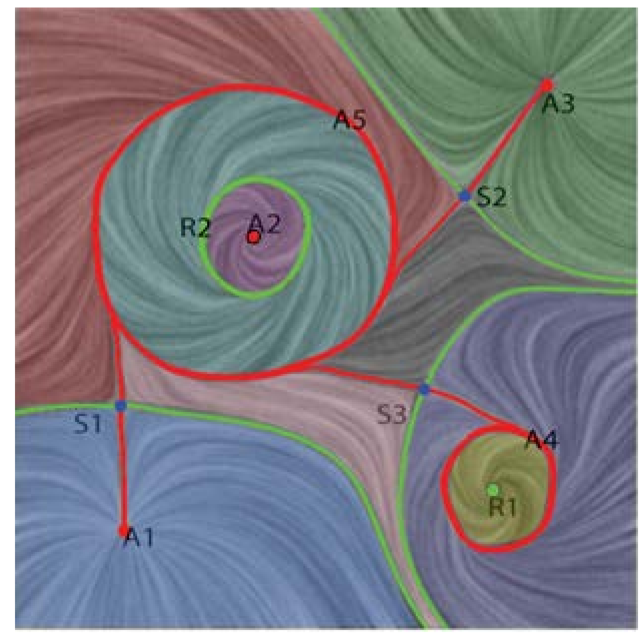


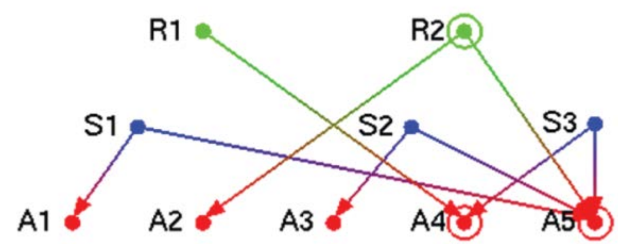


Vector Field Visualization: Computing Topology

Differential Topology Construction

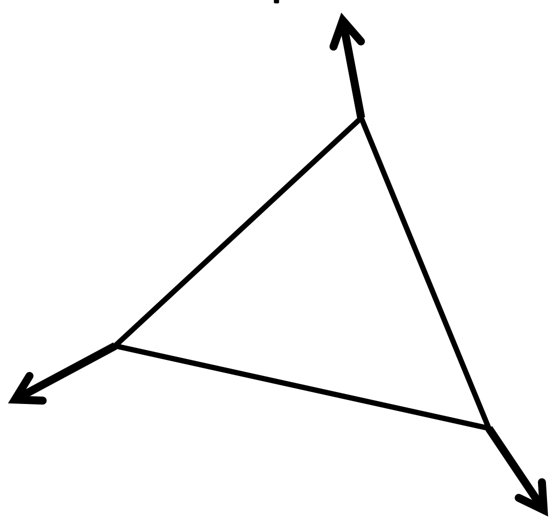
- <u>Two steps pipeline</u>
  1. Extract fixed points and periodic orbits
  2. Compute connections between these features

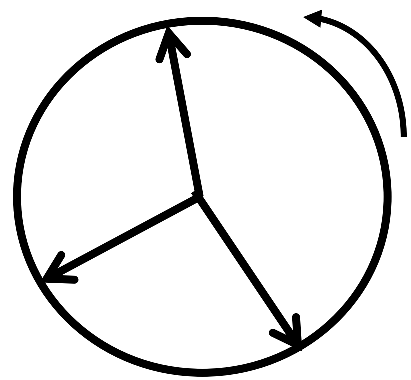




Fixed Point Extraction

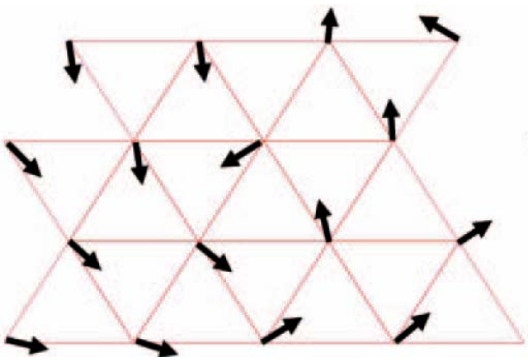
- Cell-wise
  - First, locate the cells that contain fixed points
    - Using the unique characteristics of the Poincaré index around a fixed point instead of solving a linear system

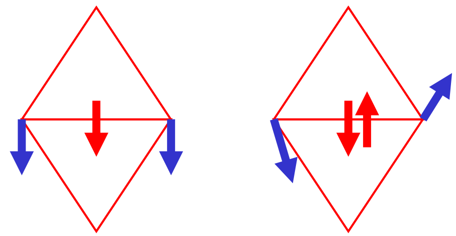


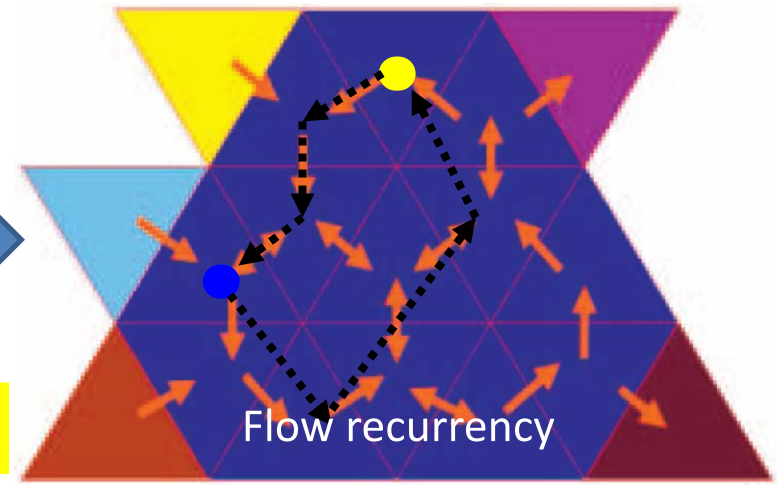


  - Second, solve for the position of the fixed point. ***How?***

Periodic Orbit Extraction

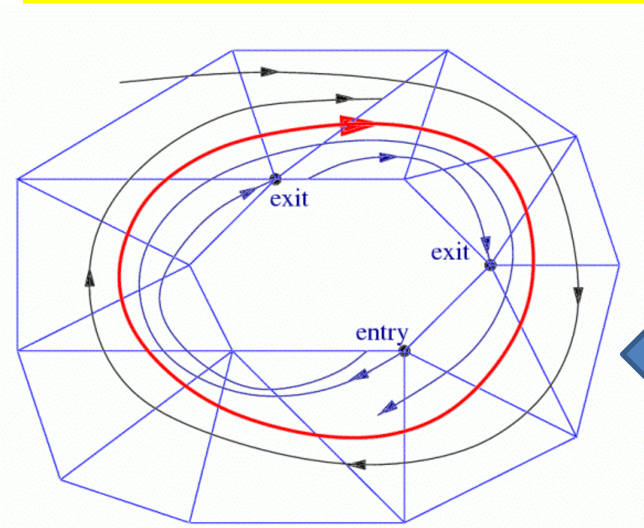


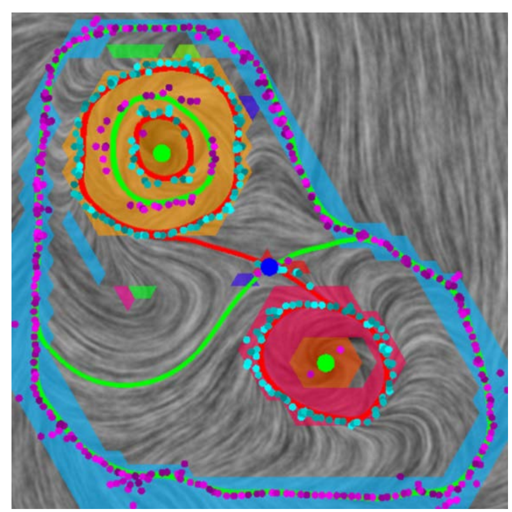


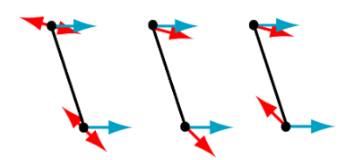


[Kalies et al. 2006]

How to quickly identify regions of flow recurrent behaviors?

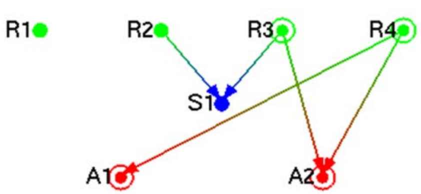


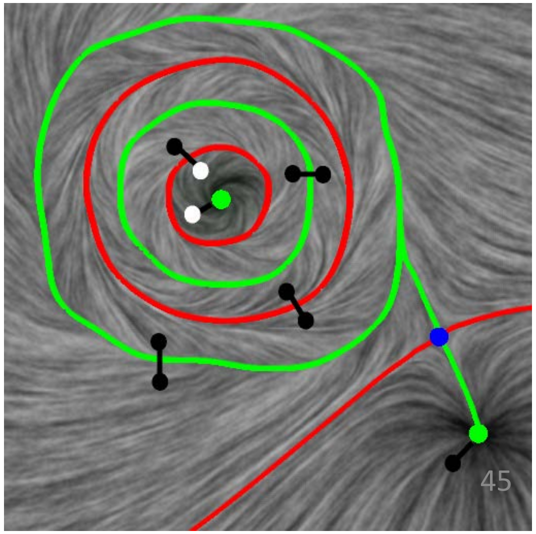




Extract Topology

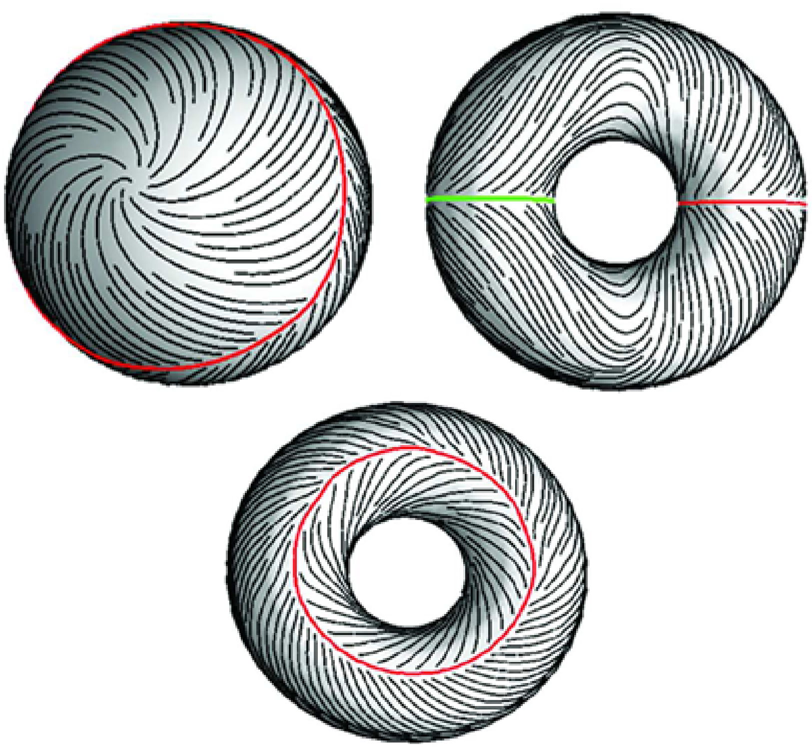
- Fixed point extraction
- Periodic orbit identification
- Compute connections
  - Separatrix computation (emitting from saddles)
  - **Other connectivity**
    - A source and a sink
    - A source/sink with a periodic orbit
    - A periodic orbit with other periodic orbit

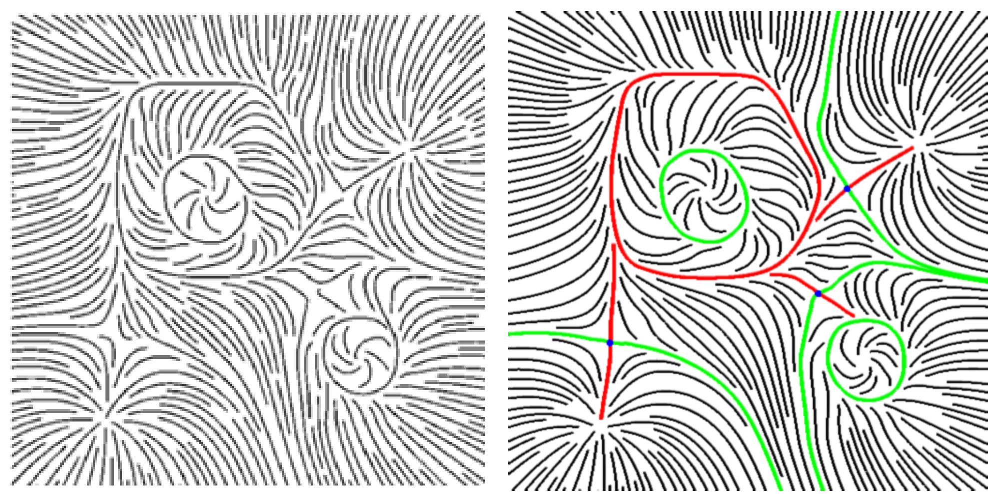




Applications (1)

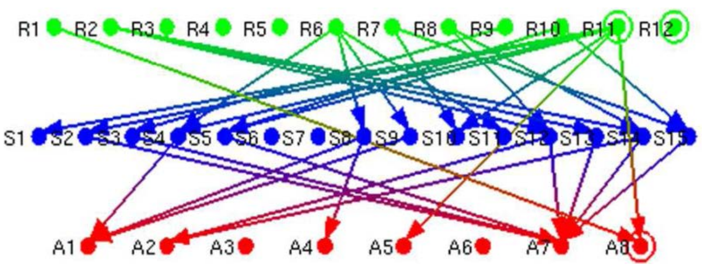
- Feature-aware streamline placement
  - First extract topology, then use it as the initial set of streamlines to compute seeds for later placement

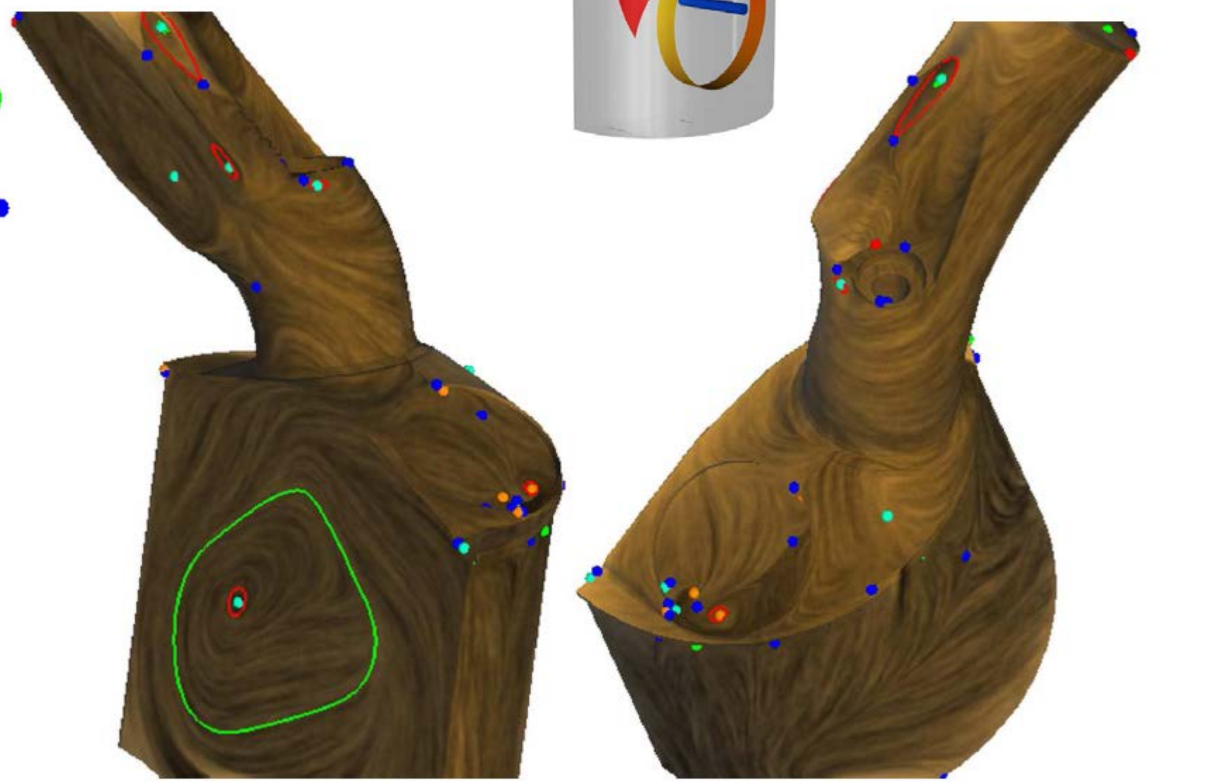




Applications (2)

- CFD simulation on gas engine
- Velocity extrapolated to the boundary





105K polygons

56 fixed points

9 periodic orbits

31.58s on analysis

Applications (3)

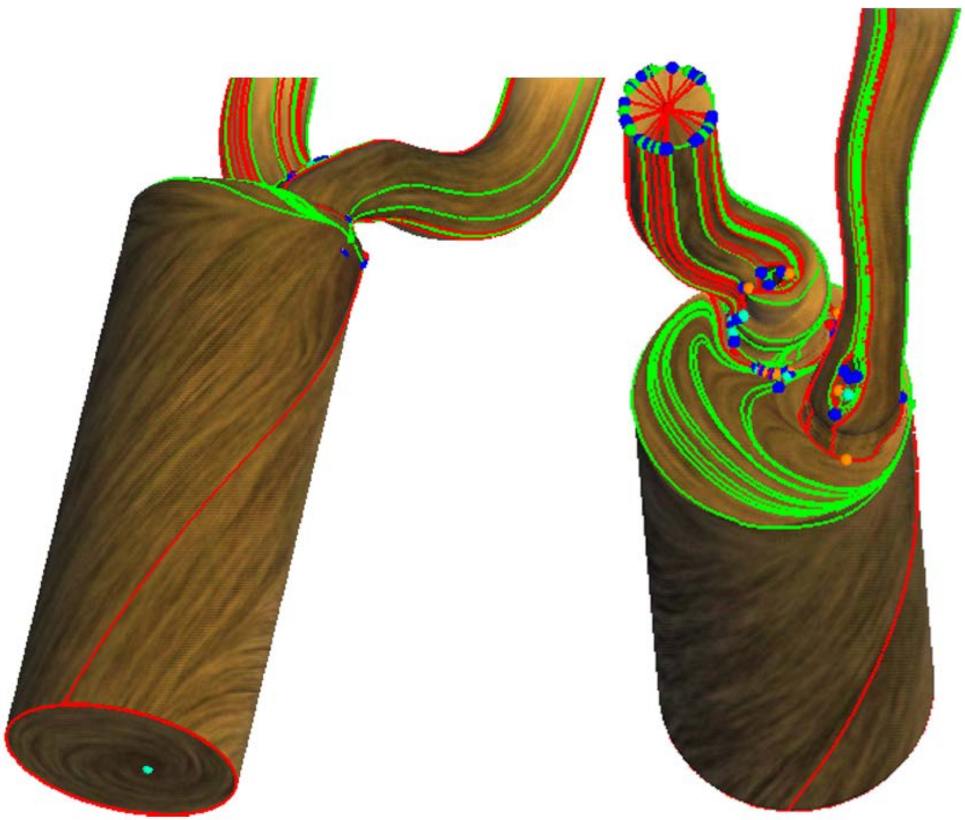
- CFD simulation on diesel engine
- Velocity extrapolated to the boundary

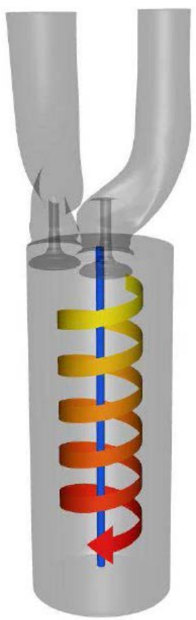
886K polygons

226 fixed points

52 periodic orbits

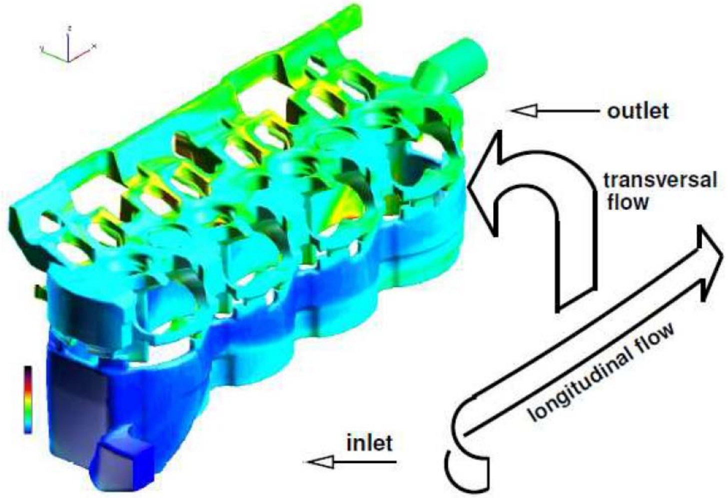
29.15s on analysis

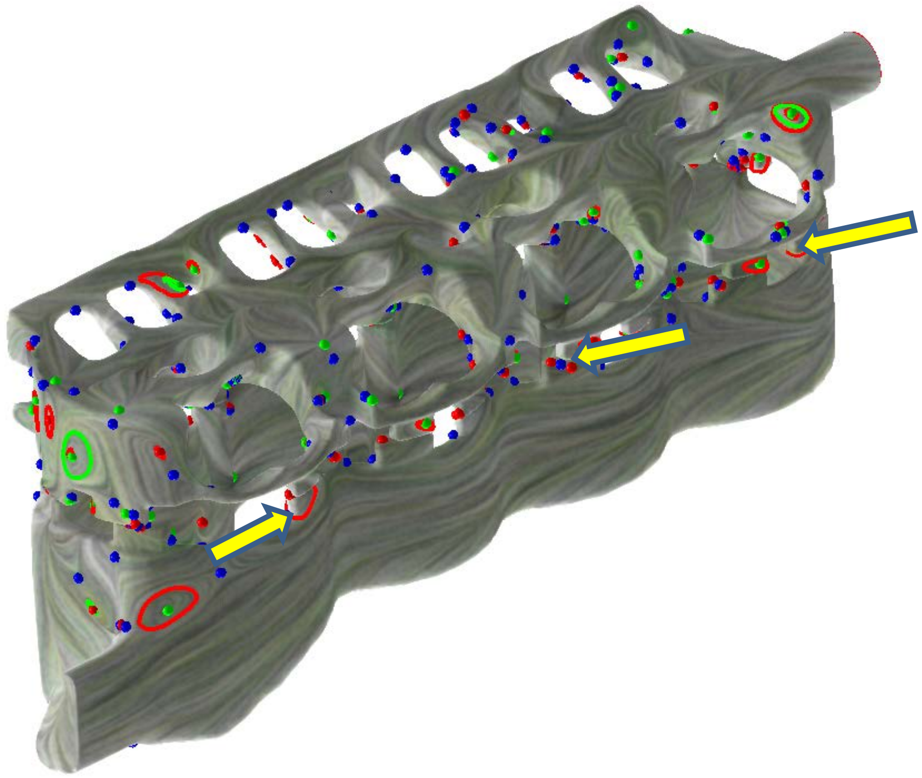




Application (4)

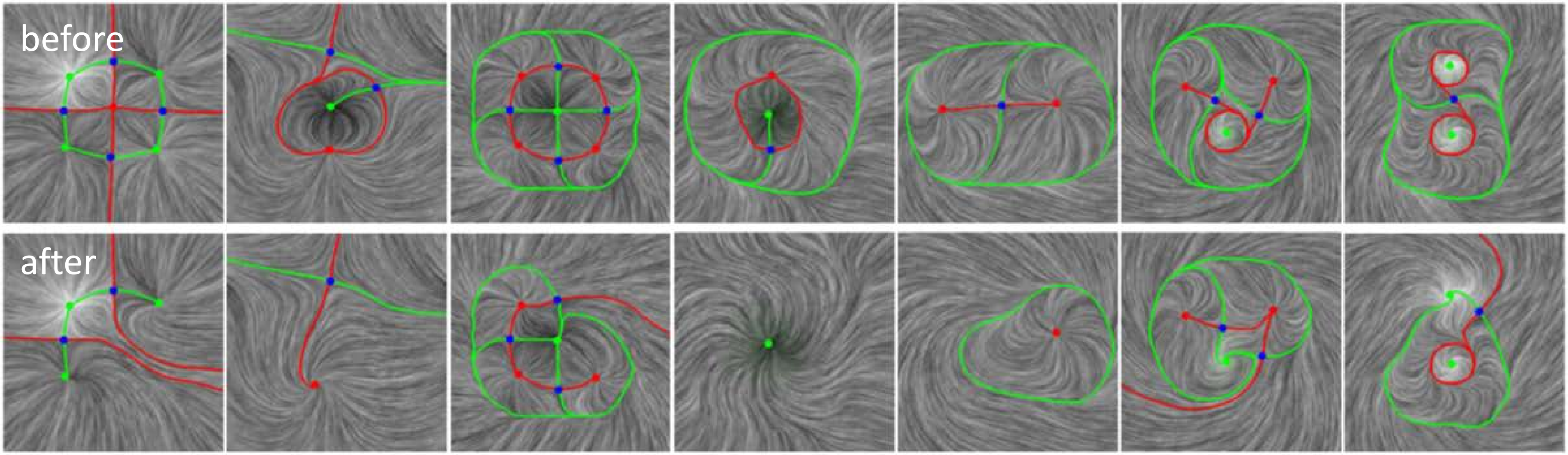
- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary



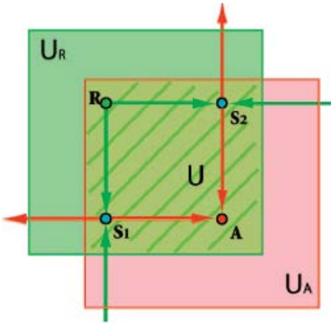


Applications – Simplification

Reduce flow complexity so that people can focus on the more important structure



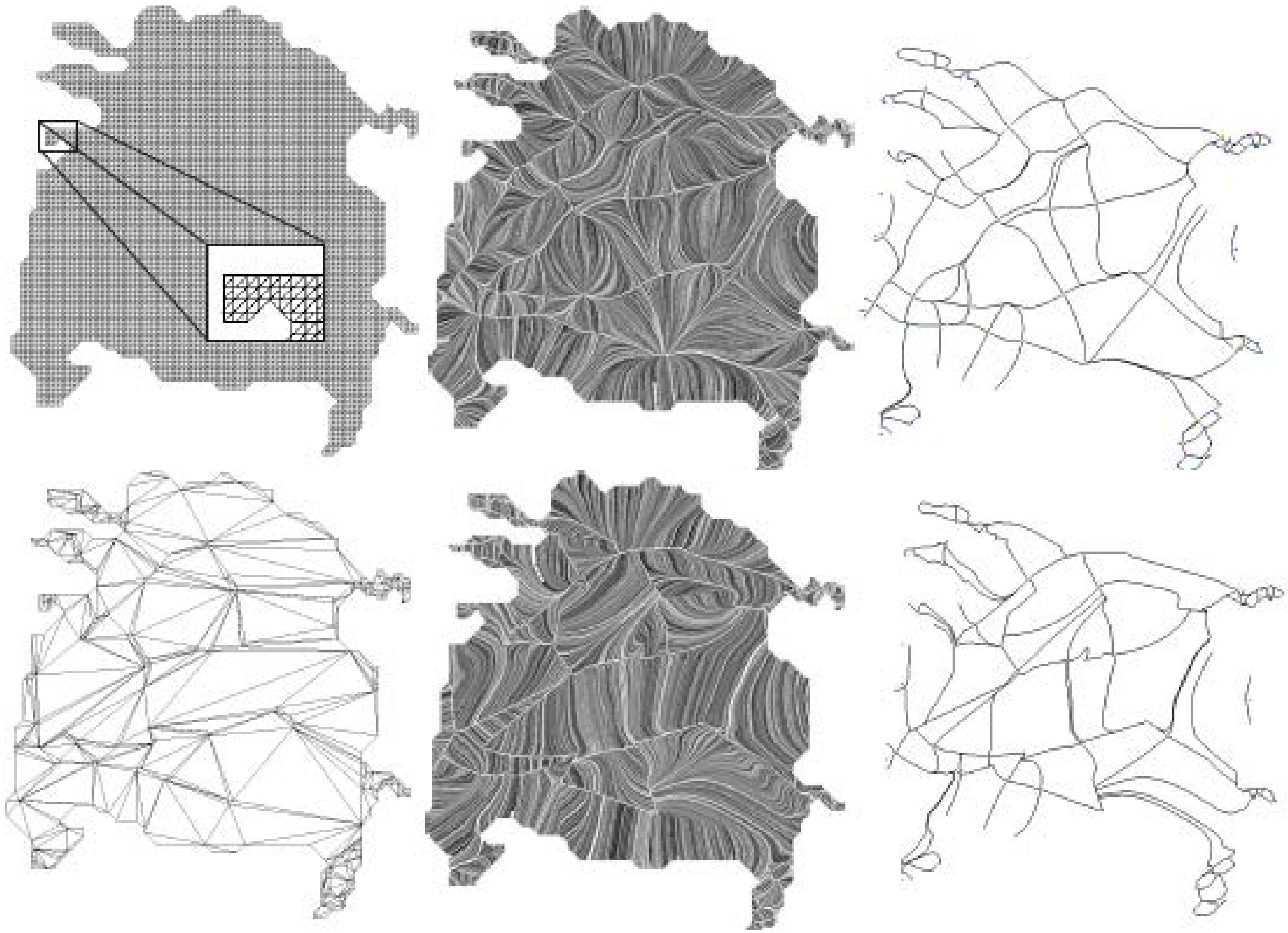
[Chen et al. 2007]



Applications – Data Compression

Before

After

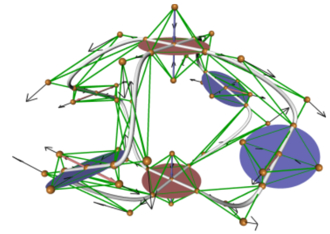


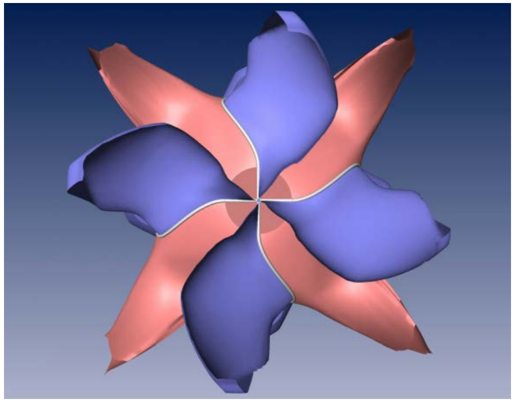
[Theisel et al. Eurographics 2003]

EXTENSION

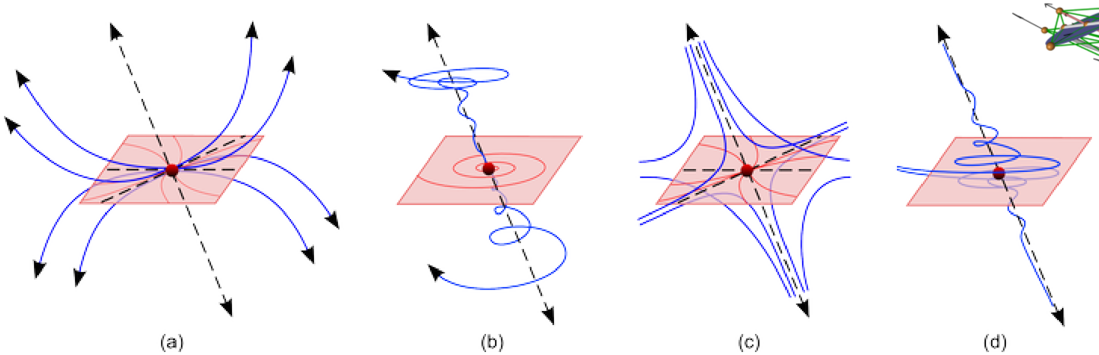
3D Flow Topology

- Fixed points



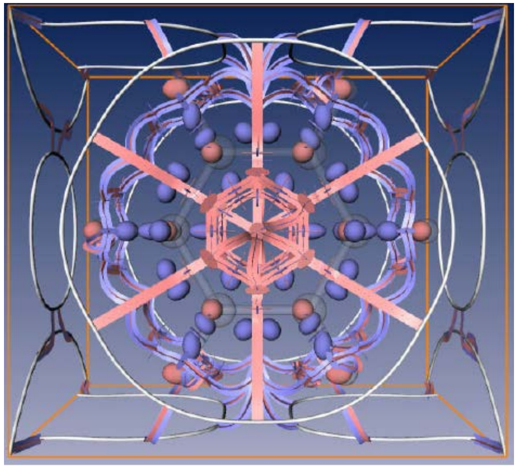


[Weinkauf et al. EG04]

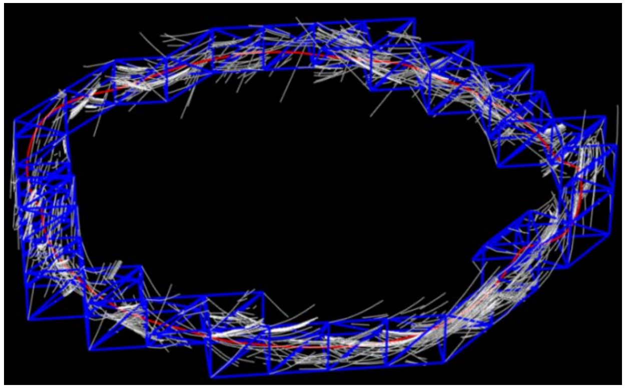


[Peikert and Sadlo http://cgg-journal.com/2010-2/02/index.html]

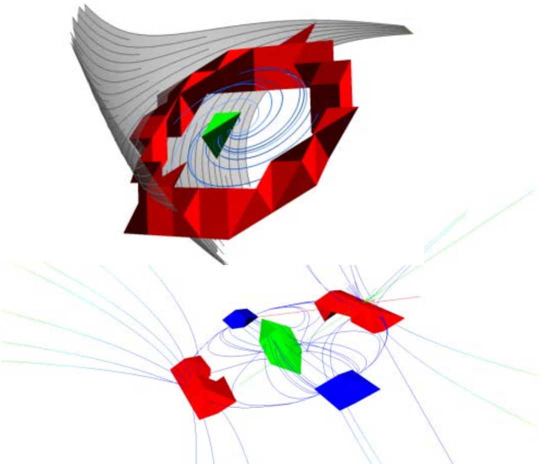
- Periodic orbits



[Weinkauf et al. VisSym 2004]



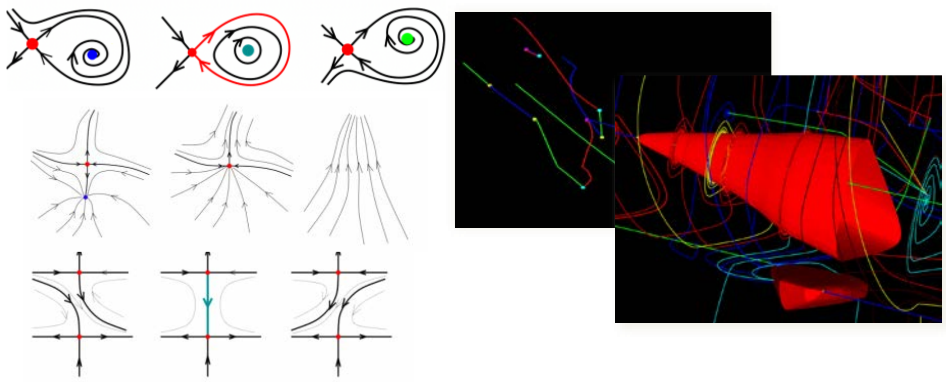
[Wischgoll and Scheuermann 2002]



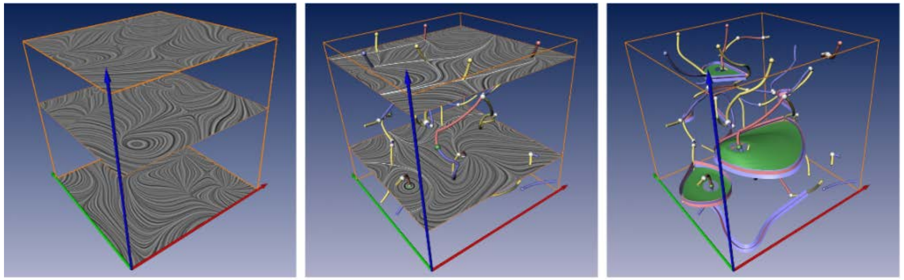
[Reich et al. TopoInVis11]

To Time-Dependent Vector Fields

- **Track the Evolution of Instantaneous Topology**



[Xavier et al. VisSym01, C&G02, Vis04]



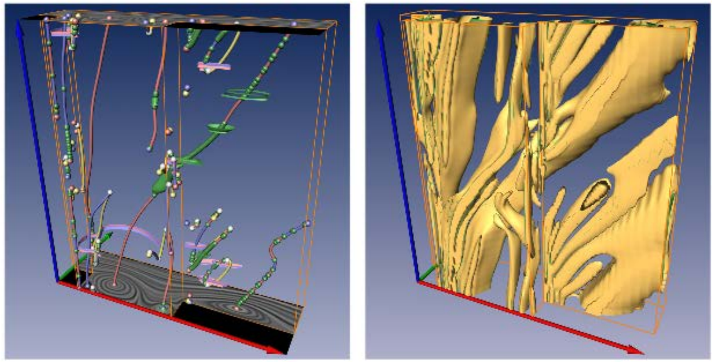
(a) LIC images at 3 different time slices.

(b) Tracking the locations of critical points as stream lines (red/blue/yellow); local bifurcations: Hopf bifurcations (green spheres), fold bifurcations (gray spheres).

(c) Global bifurcations: saddle connections (red/blue flow ribbons), tracked closed stream lines (green surfaces).

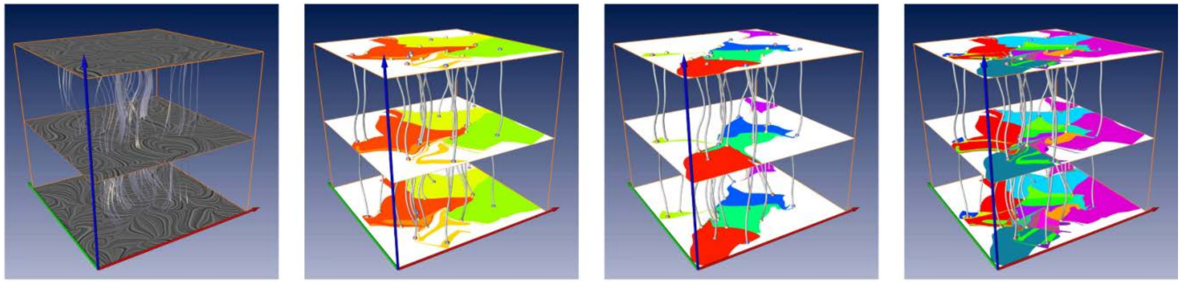
[Theisel et al. VisSym2003, Vis04, TVCG05]

- **Pathline-based**



(b) Stream line oriented topology of the first 100 time steps. (c) Path line oriented topology of the first 100 time steps.

[Theisel et al. Vis04, TVCG05]



(a) The vector field **p**.

(b) Critical path lines and basins for forward integration.

(c) Critical path lines and basins for backward integration.

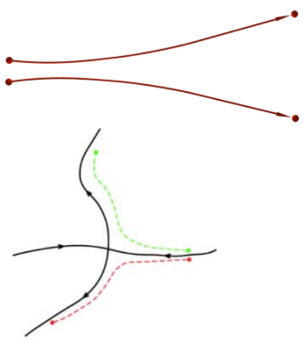
(d) Overlayed basins for forward and backward integration.

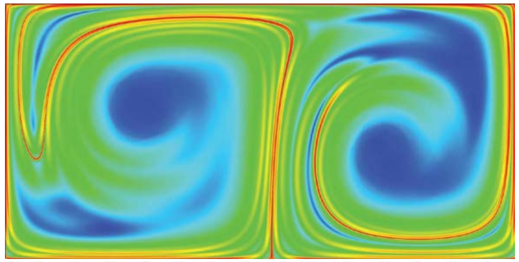
[Shi et al. EuroVis06]

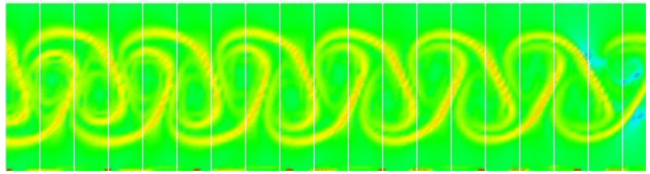
To Time-Dependent Vector Fields

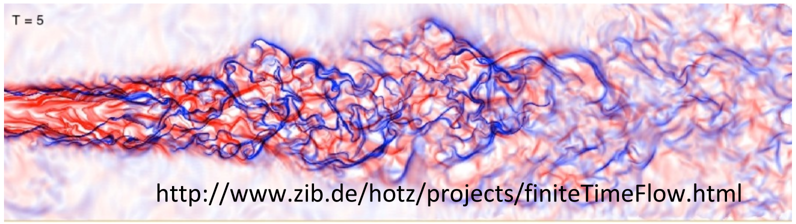
- **FTLE**

[Haller 2001, Shadden et al. 2005, Garth et al. CGF08, Garth et al. Vis07, Lekien et al. 2007, Sadlo and Peikert TVCG07, Fuchs et al.PG10 etc., Kuhn et al. PacificVis12, etc…]

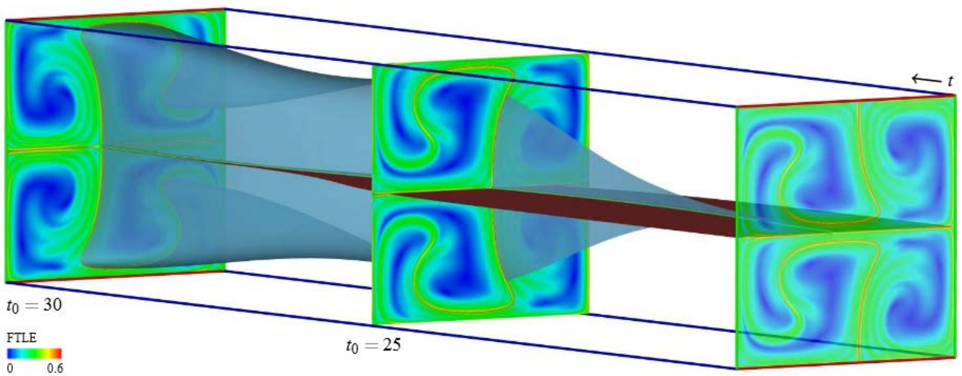




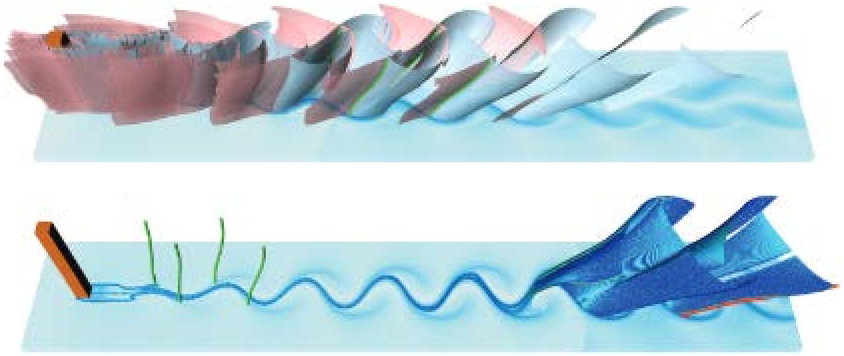




- **Streaklines/Streak-surface based**

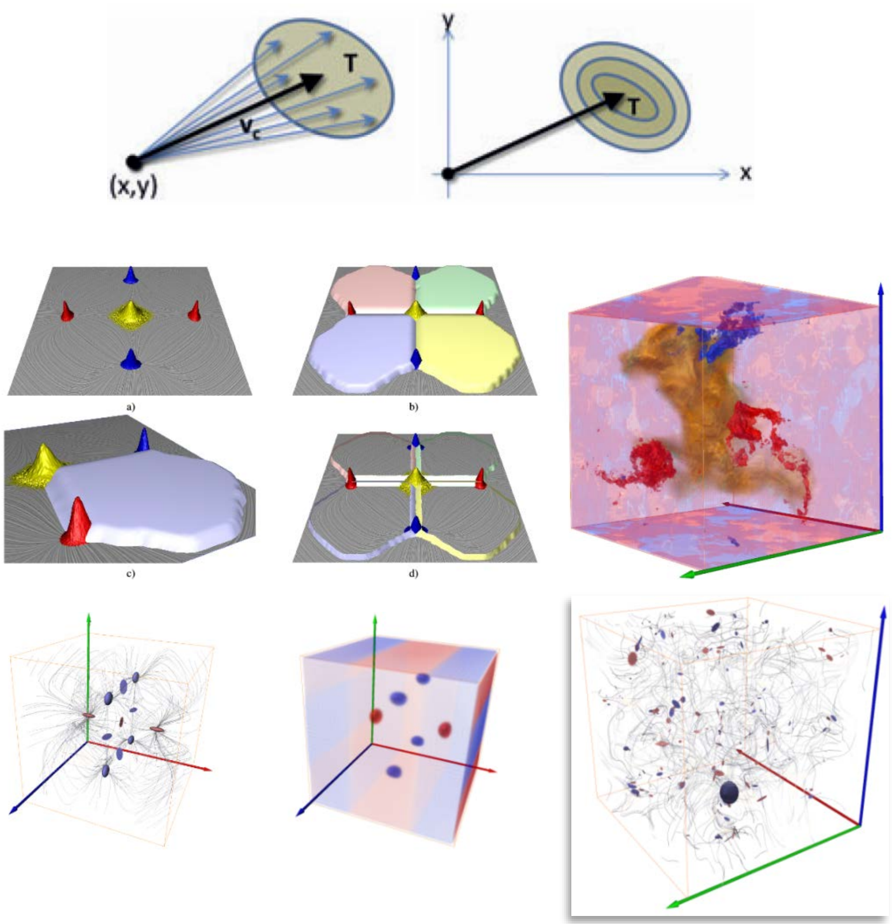


[Sadlo and Weiskopf EG11]

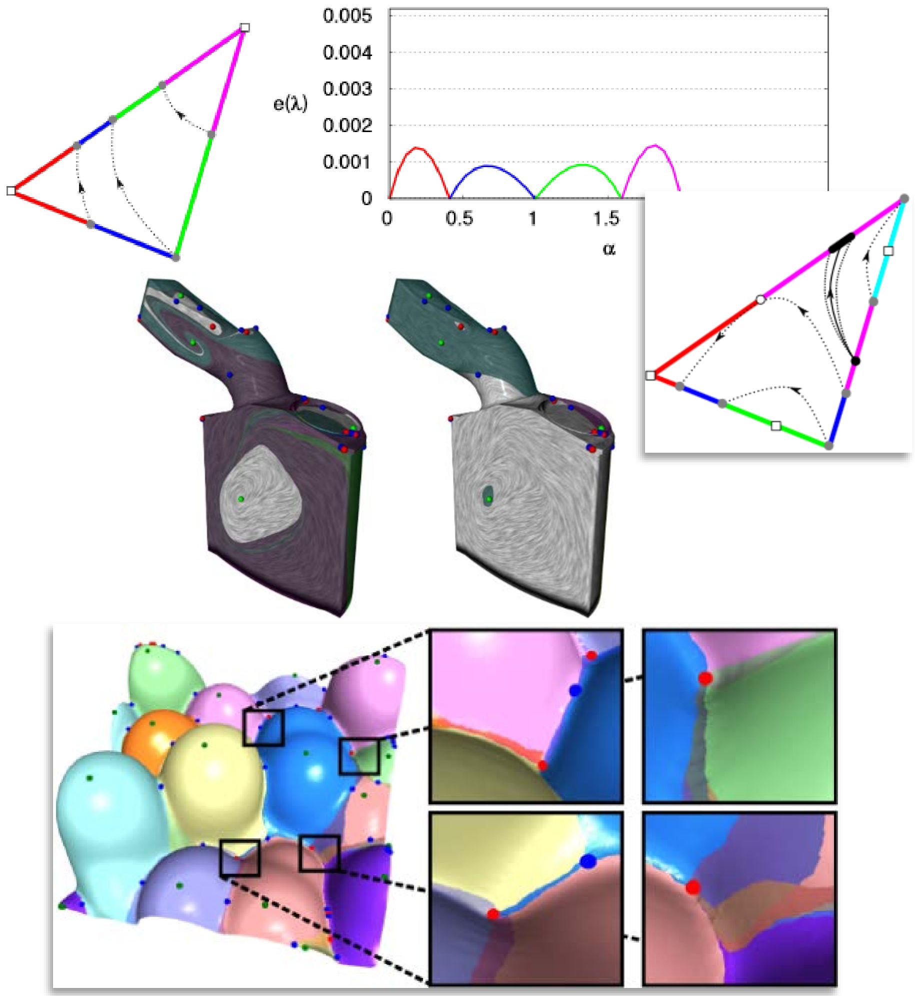


[Uffinger et al. TVCG13]

To Uncertainty Vector Fields



[Otto et al. EG10, PacificVis11]



[Bhatia et al. PacificVis11, TVCG2012]

Additional Reading

- Frits H. Post, Benjamin Vrolijk, Helwig Hauser, Robert S. Laramee and Helmut Doleisch. **The State of the Art in Flow Visualization: Feature Extraction and Tracking**. Computer Graphics Forum, 22 (4): pp. 1-17, 2003.

- Robert S. Laramee, Helwig Hauser, Lingxiao Zhao, , and Frits H. Post. Topology-based flow visualization, the state of the art. In H. Hagen H. Hauser and H. Theisel, editors, Topology-Based Methods in Visualization 2005, Mathematics and Visualization, pages 1–19. Springer-Verlag, 2007.

- Tobias Salzbrunn, Thomas Wischgoll, Heike Jänicke, and Gerik Scheuermann. The state of the art in flow visualization: Partition-based techniques. In H. Hauser, S. Strassburger, and H. Theisel, editors, In Simulation and Visualization 2008 Proceedings, pages 75–92. SCS Publishing House, 2008.

- Armin Pobitzer, Ronald Peikert, Raphael Fuchs, Benjamin Schindler, Alexander Kuhn, Holger Theisel, Kresimir Matkovic, and Helwig Hauser. The state of the art in topology-based visualization of unsteady flow. Computer Graphics Forum, 30(6):1789–1811, September 2011.

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