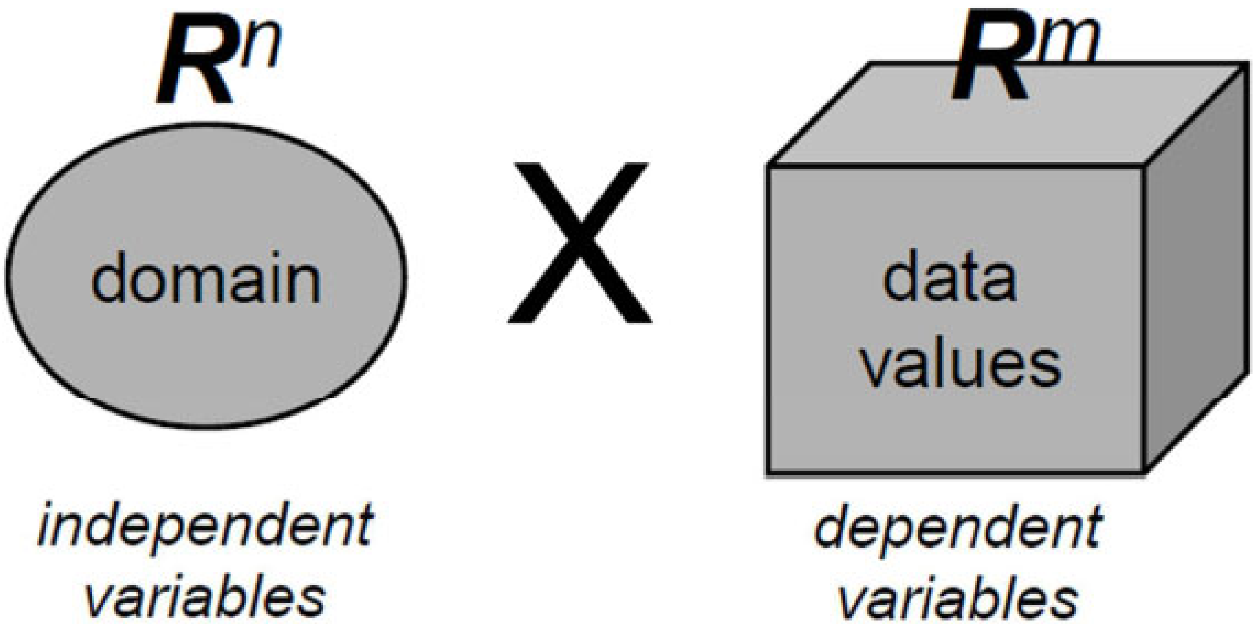
What we have learned so far…

- **Scalar field** visualization
  - Describe scalar *quantities* (**0D**)
  - **Methods**: direct, geometric-based, feature-based

- **Vector field** visualization
  - Describe *directional* information (**1D**)
  - **Methods**: direct, geometric-based, <u>*texture-based*</u>, feature-based



scientific data $\subseteq \mathbf{R}^{n+m}$

Source: VIS, University of Stuttgart

1D	scalar
2D	vector
3D	tensor
+time	

Tensor Field Visualization Introduction

Goal: understand important concepts of tensors and their basic processing

One Simple Example of Tensor

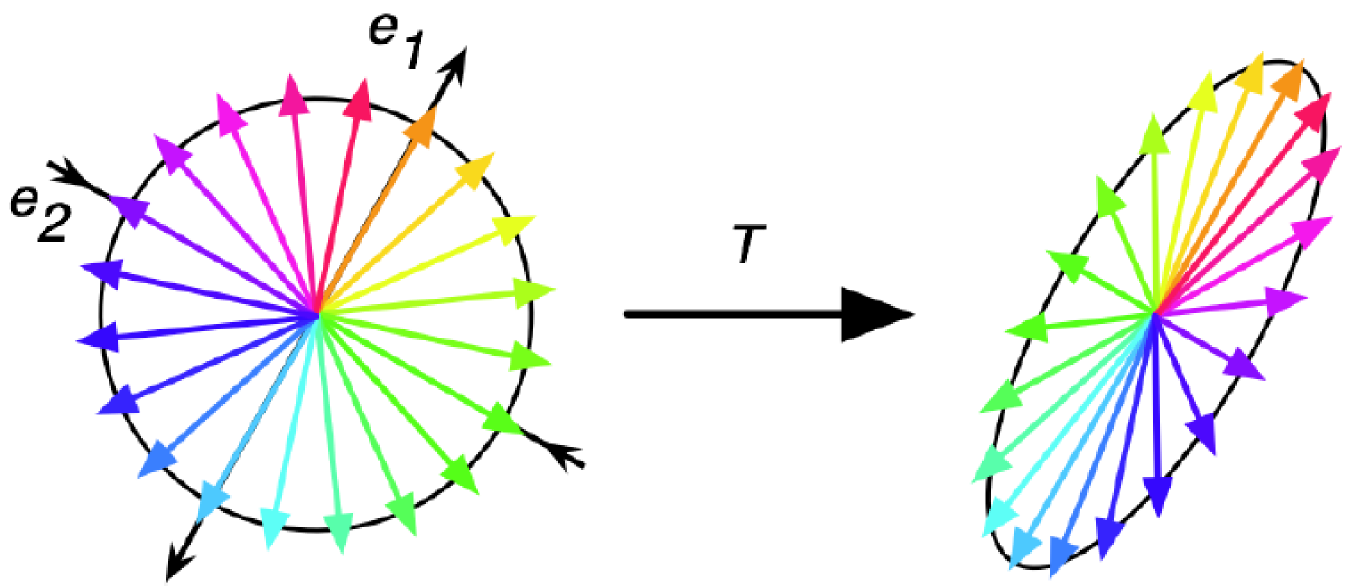
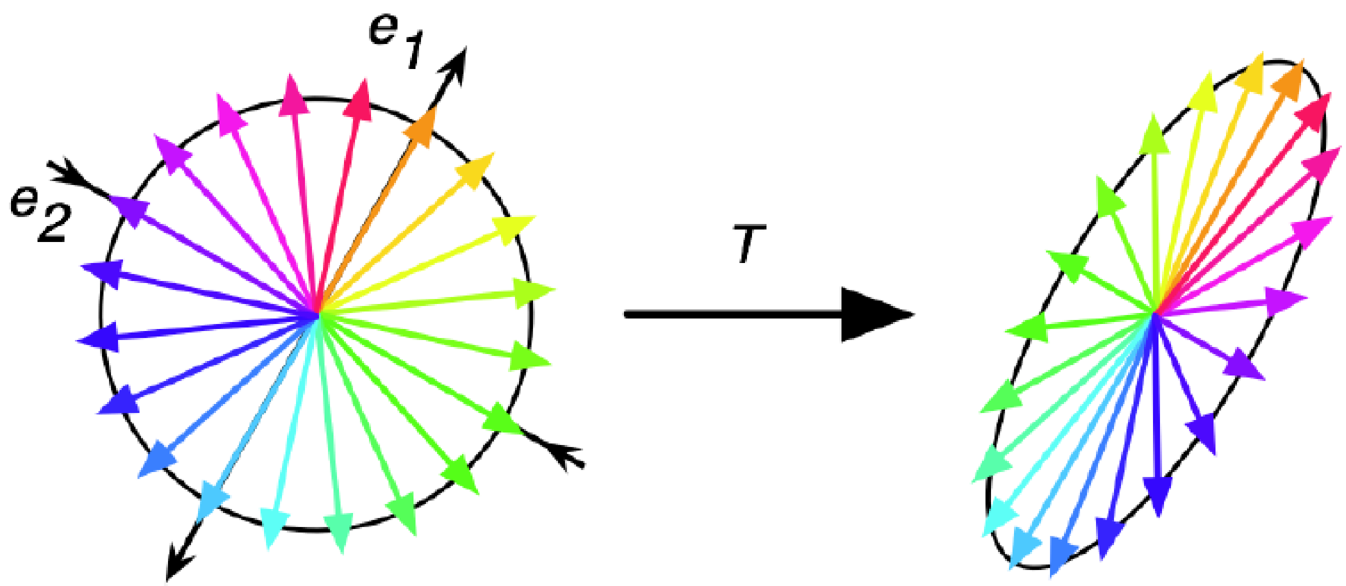


Illustration of a symmetric second-order tensor as a **deformation** tensor. The tensor is uniquely determined by its action on all unit vectors, represented by the circle in the left image. The eigenvector directions are highlighted as black arrows.
In this example **one eigenvalue (λ2) is negative**. Therefore, all vectors are mirrored at the axis spanned by eigenvector ***e*1**. The eigenvectors are the directions with strongest **normal** deformation.

One Simple Example of Tensor



Tensor describes certain ***higher-order*** property of the space that both scalar and vector-valued cannot

Definition

- A **second-order tensor** T is defined as a **bilinear function** from two copies of a vector space V into the space of real numbers

$$T: V \times V \to R$$

inner product

Definition

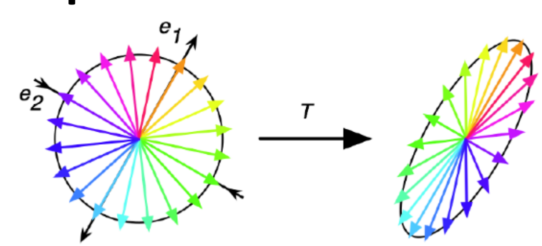
- A **second-order tensor** T is defined as a **bilinear function** from two copies of a vector space V into the space of real numbers

$$T: V \times V \rightarrow R$$

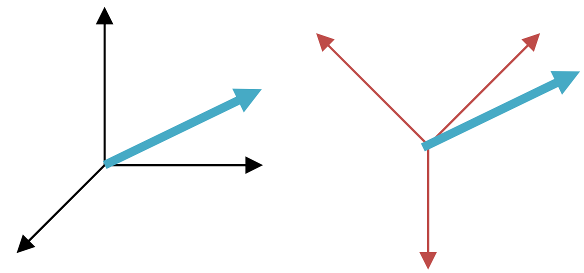
- Or: a **second-order** tensor T as a **linear operator** that maps any vector v ∈V onto another vector w ∈ V

$$T: V \rightarrow V$$

- The definition of a tensor as a linear operator is prevalent in physics.



Definition



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- In this case, the tensor is uniquely defined by its components (entries) and is represented as a matrix.

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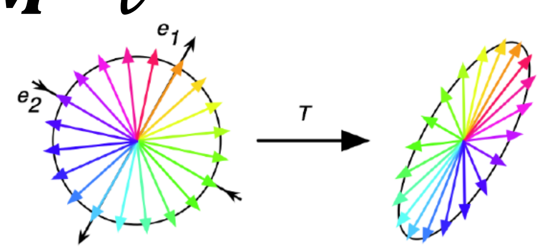
- Considering definition (1), we have

$$T(v, w) = w^T \cdot \boldsymbol{M} \cdot v \qquad \forall v, w \in V$$

where $v = v_1 b_1 + \cdots + v_n\, b_n, w = w_1 b_1 + \cdots + w_n\, b_n$

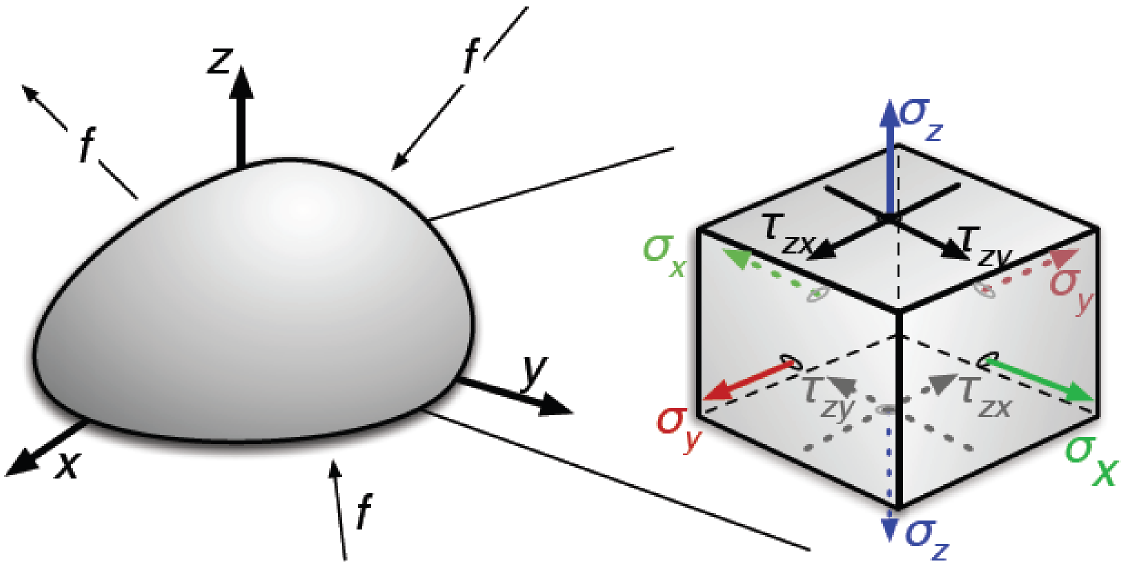
- For definition (2), we have $w = T(v) = \boldsymbol{M} \cdot v$



Applications

- Tensors describe entities that scalars and vectors cannot describe sufficiently.
  - continuum mechanics,
  - medicine,
  - geology,
  - astrophysics,
  - architecture,
  - and *many more*

Tensors in Mechanical Engineering



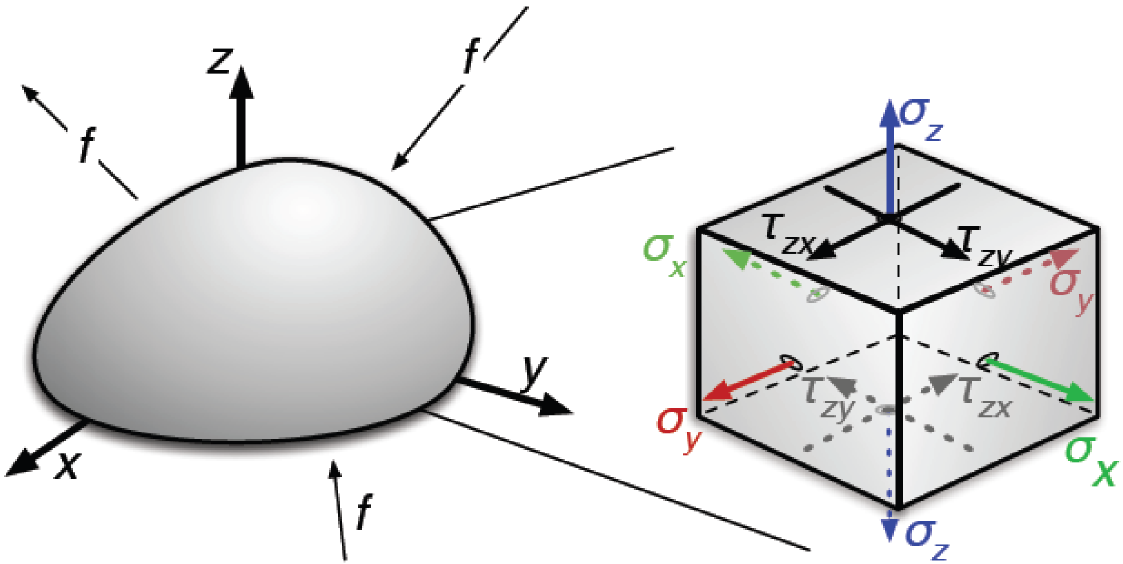
(a)

$$M = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

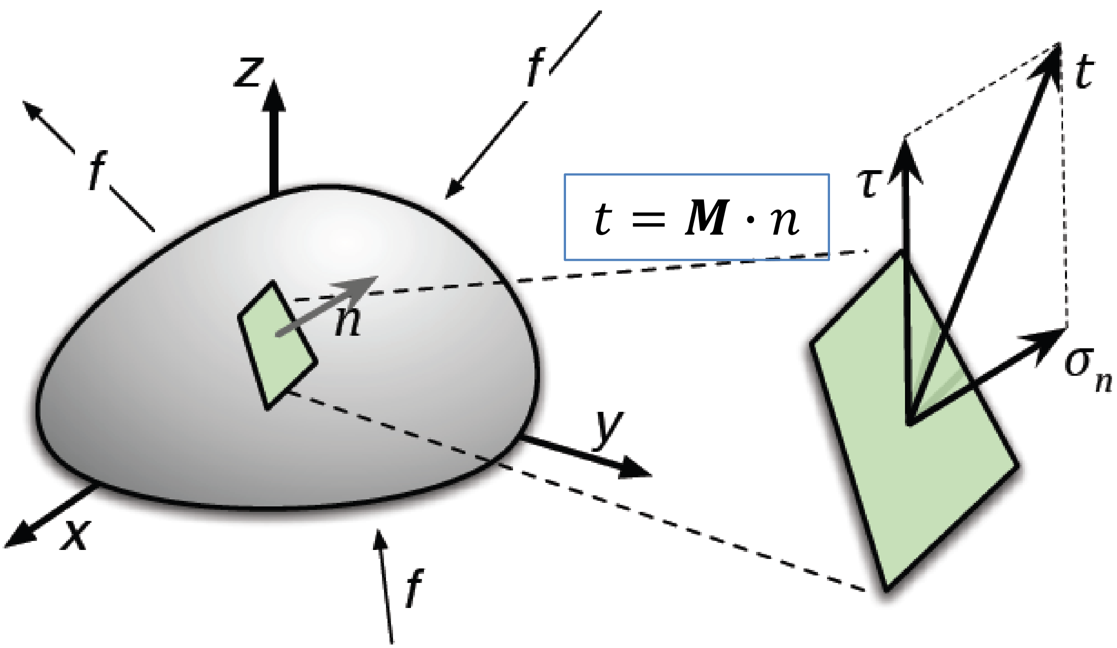
Stress tensors describe internal forces or stresses that act within deformable bodies as reaction to **external** forces

(a) External forces f are applied to a deformable body. **Reacting forces** are described by a three-dimensional stress tensor that is composed of three normal stresses σ and three shear stresses τ.

Tensors in Mechanical Engineering



(a)



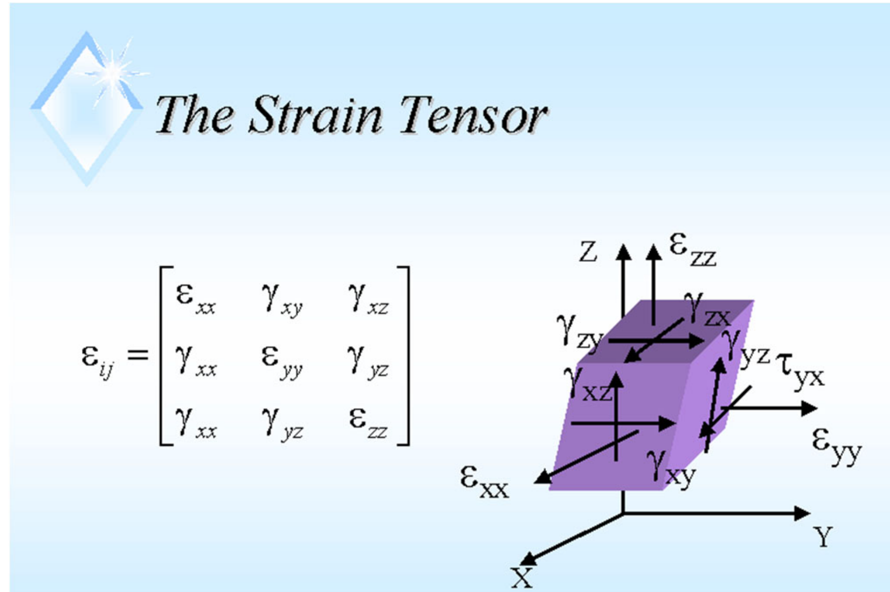
Stress tensors describe internal forces or stresses that act within deformable bodies as reaction to external forces

(a) External forces f are applied to a deformable body. Reacting forces are described by a three-dimensional stress tensor that is composed of three normal stresses σ and three shear stresses τ.

(b) Given a surface normal n of some cutting plane, the stress tensor maps *n* to the **traction vector** *t*, which describes the internal forces that act on this plane (normal and shear stresses).

Tensors in Mechanical Engineering

- Strain tensor - related to the deformation of a body due to stress by the material's constitutive behavior.



http://enpub.fulton.asu.edu/concrete/elasticity2_95/sld006.htm

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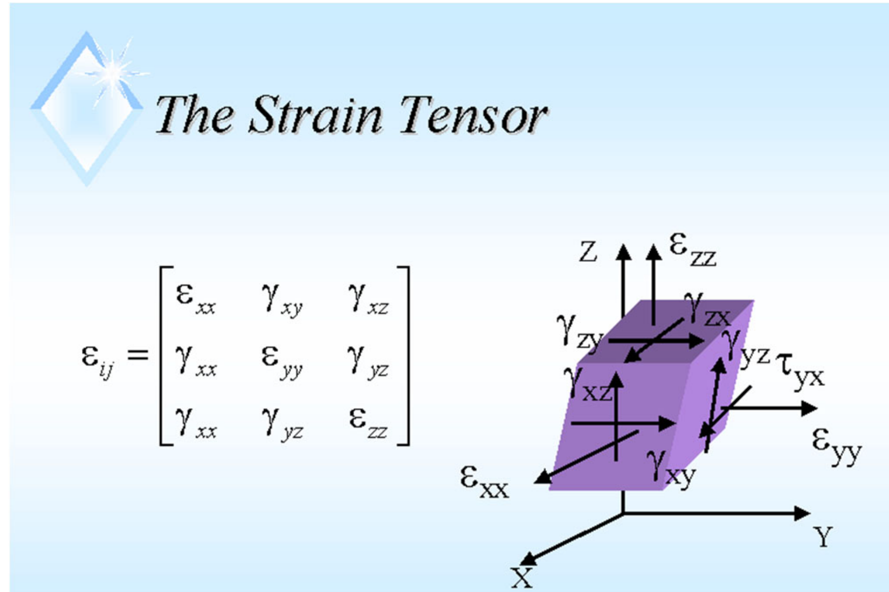


http://enpub.fulton.asu.edu/concrete/elasticity2_95/sld006.htm

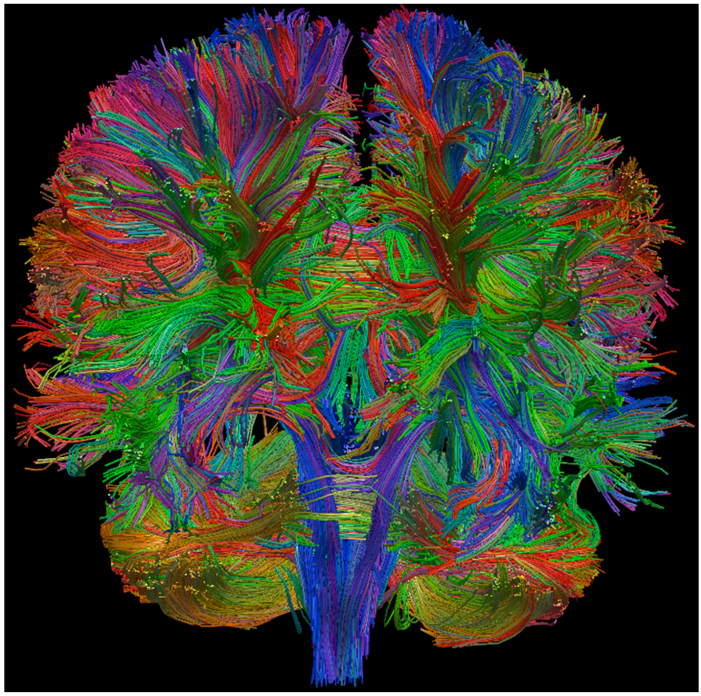
- **Deformation gradient tensor** – gradient of displacements of material points (*think of Jacobian*)

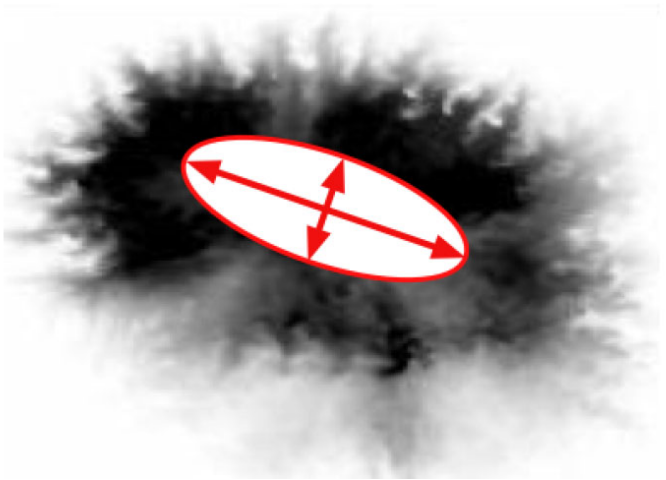
- The strain tensor is a normalized measure based on the deformation gradient tensor (***the symmetric part of the Jacobian***)

$$\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Diffusion Tensor Imaging (DTI)

- For medical applications, diffusion tensors describe the anisotropic diffusion behavior of water molecules in tissue.
- Here, the molecule motion is driven by the Brownian motion and not the concentration gradient.





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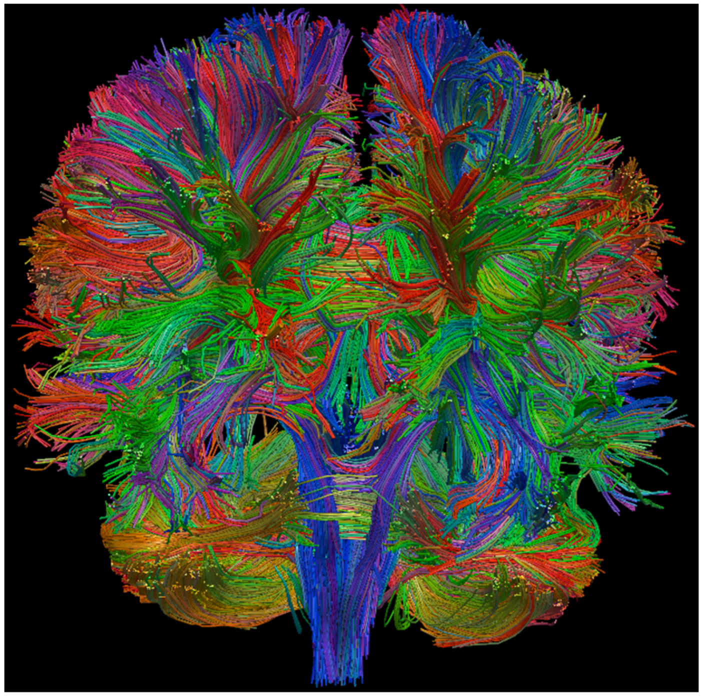
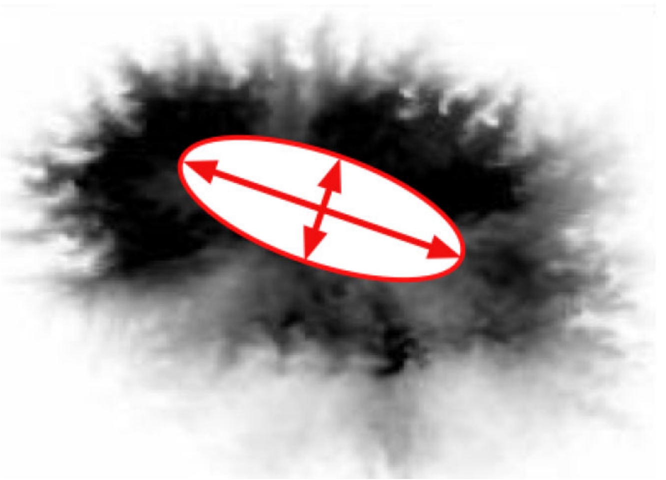


- For medical applications, diffusion tensors describe the anisotropic diffusion behavior of water molecules in tissue.
- Here, the molecule motion is driven by the Brownian motion and not the concentration gradient.

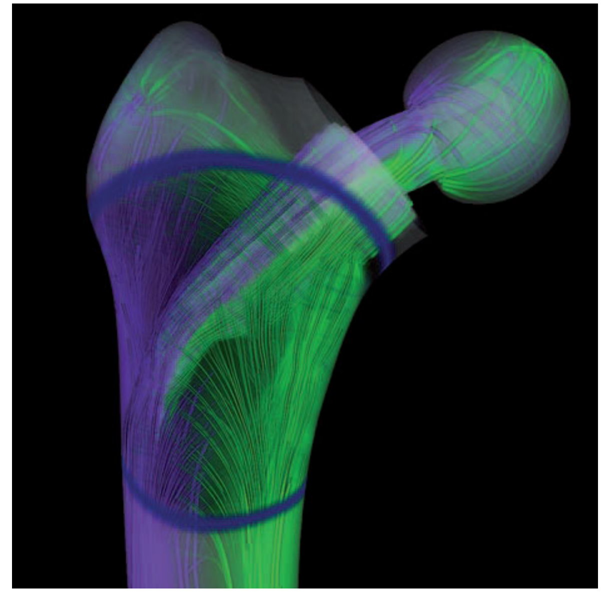
- The tensor contains the following information about the diffusion: **its strength depending on the direction and its anisotropy**
- It is **positive semi-definite** and symmetric.



Note that in practice the positive definiteness of diffusion tensors can be violated due to measurement noise.

Tensors in Medicine

- Diffusion tensors are not the only type of tensor that occur in the medical context.
- In the context of implant design, stress tensors result from simulations of an implant's impact on the distribution of physiological stress inside a bone.



[DICK et al. Vis09]

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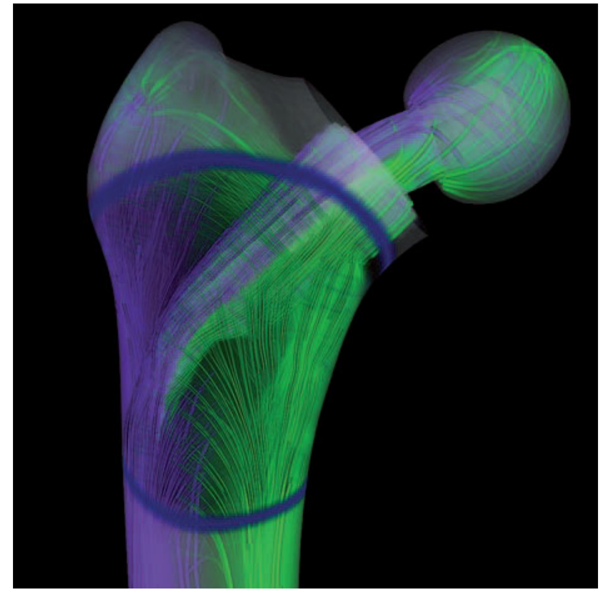
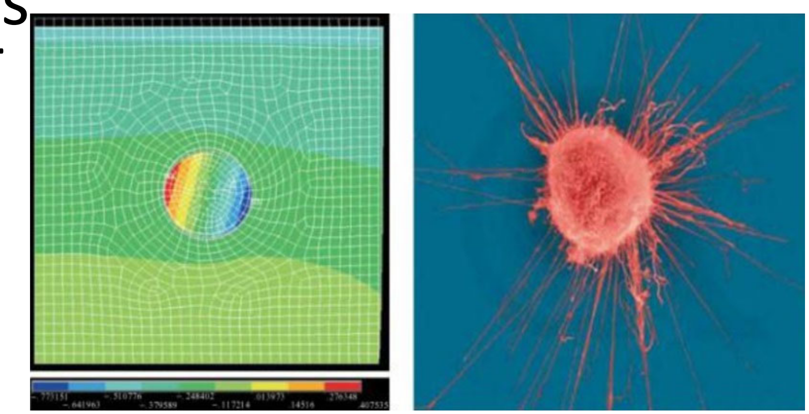


[DICK et al. Vis09]

- An application related to strain tensors is used in elastography where MRI, CT or ultrasound is used to measure elastic properties of soft tissues. Changes in the elastic properties of tissues can be an important hint to cancer or other diseases

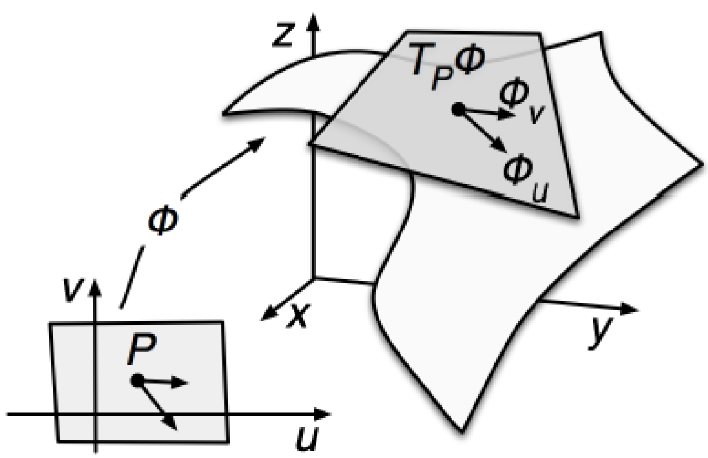


[SOSA-CABRERA et al., 2009]

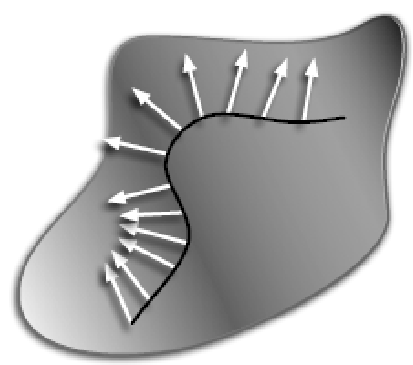
Tensors in Geometry

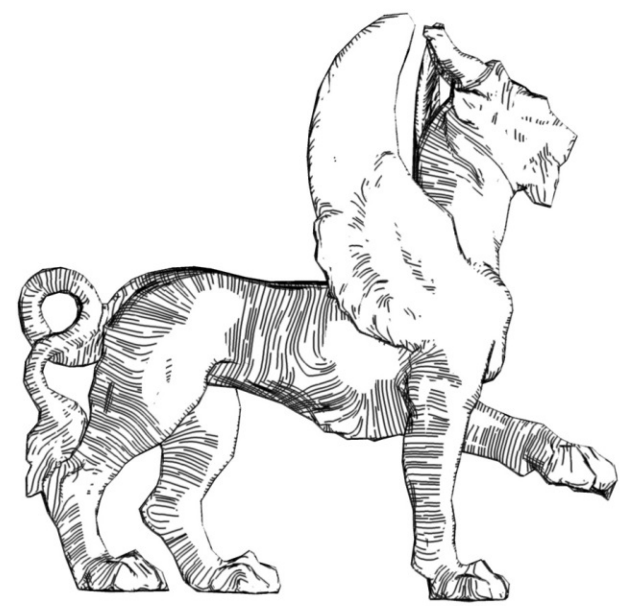
- Curvature tensors - change of surface normal in any given direction
- Metric tensors - relates a direction to distances and angles; defines how angles and the lengths of vectors are measured independently of the chosen reference frame

$$T(v, w) = w^T \cdot M \cdot v$$



(a)

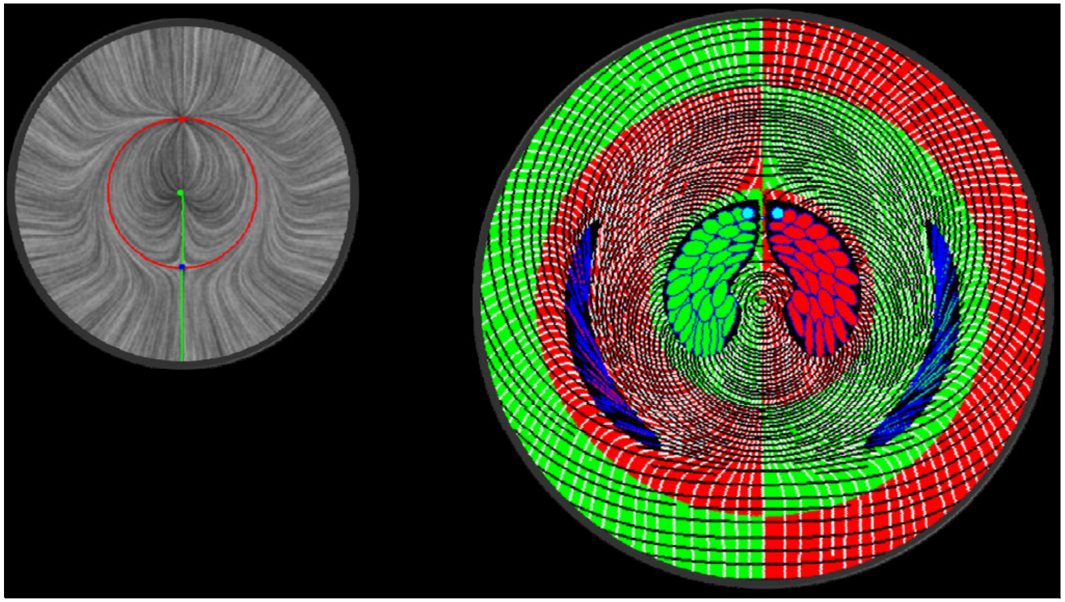

(b)

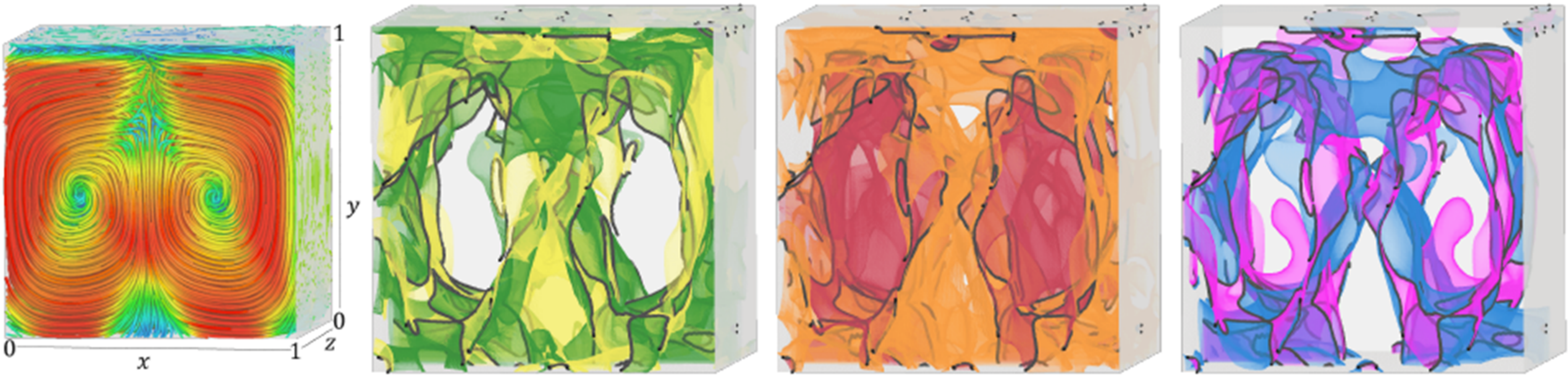


Gradient Tensor of Velocity Field

$$\nabla V = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{bmatrix}$$

Asymmetric tensors!

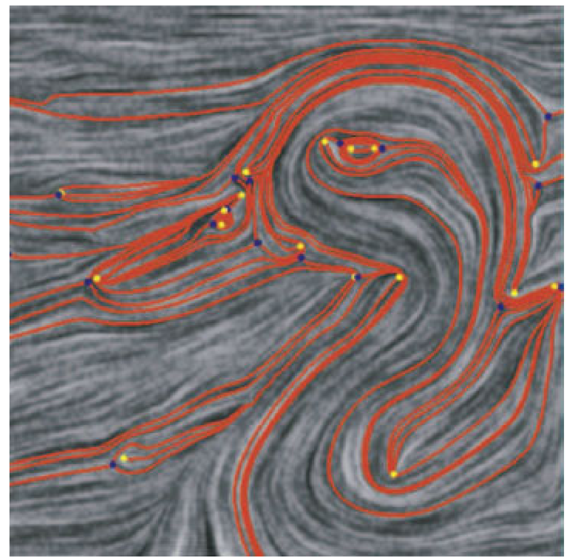




(a) vector field (b) degenerate surfaces (c) balanced surfaces (d) neutral surfaces

Tensors in Image Analysis

- Image analysis



- Computer vision

[Zhang et al, TVCG2007]

Some Math of Tensors

Definition (Recall)

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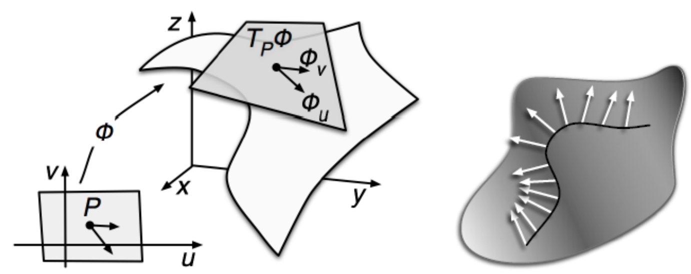
$$T: V \times V \rightarrow R$$



(a) (b)

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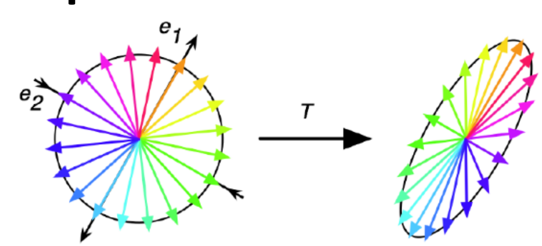
- The definition of a tensor as a linear operator is prevalent in physics.



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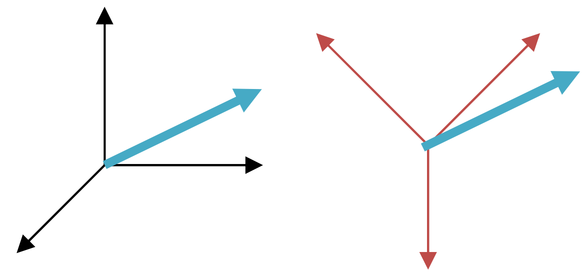

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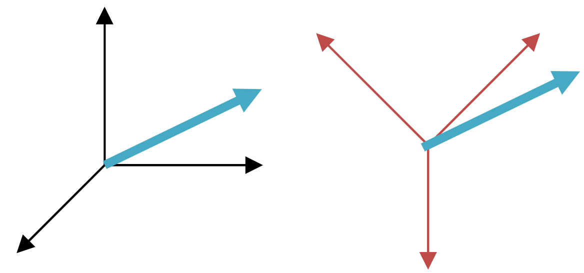
where $v = v_1 b_1 + \cdots + v_n b_n$, $w = w_1 b_1 + \cdots + w_n b_n$

- For(2), we have $T(v) = \boldsymbol{M} \cdot v$

Tensor Invariance



- Some tensor properties are independent of specific reference frames, i.e., they are invariant under coordinate transformations.

- Invariance qualifies tensors to describe physical processes independent of the coordinate system.

- More precisely, the *tensor components (entries in matrix) change according to the transformation into another basis*; **the characteristics (which ones?) of the tensor are preserved**. Consequently, tensors can be analyzed using any convenient reference frame.

- ***Rotational invariant***

Tensor Diagonalization

- The tensor representation becomes especially simple if it can be diagonalized.

- The complete transformation of T from an arbitrary basis into the eigenvector basis, is given by

$$UTU^T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

- The diagonal elements $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues and U is the orthogonal matrix that is composed of the eigenvectors, that is $(e_1,$ $e_2, e_3)$

- The diagonalization generally is computed numerically via singular value decomposition (**SVD**).

Tensor Properties

- **Symmetric Tensors**. A tensor S is called symmetric if it is invariant under permutations of its arguments

$$S(v,w) = S(w,v) \quad \forall v,w \in V$$

$$T_{ij} = T_{ji}, \text{ third-order } T_{ijk} = T_{jik} = T_{ikj} = T_{kij} = \cdots$$

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- **Antisymmetric Tensors**. A tensor A is called antisymmetric or skew-symmetric if the sign flips when two adjacent arguments are exchanged

$$A(v, w) = -A(w, v) \quad \forall v, w \in V$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

what is the implication of a and d?

Tensor Properties

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$$A(v,w) = -A(w,v) \quad \forall v,w \in V$$

- **Traceless Tensors**. Tensors T with zero trace, i.e., $tr(T) \ = \sum_{i=0}^{n-1} T_{ii}$, are called traceless.

Tensor Properties

- **Positive (Semi-) Definite Tensors**. A tensor T is called positive (semi-) definite if

$$T(v, v) > (\geq) 0 \qquad v^T \cdot \boldsymbol{M} \cdot v > 0 \;\; (v \neq 0)$$

Their eigenvalues and their determinant are greater than zero.

Tensor Properties

- **Positive (Semi-) Definite Tensors**. A tensor T is called positive (semi-) definite if

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Their eigenvalues and their determinant are greater than zero.

- **Negative (Semi-) Definite Tensors**. A tensor T is called negative (semi-) definite if

$$T(v,v) < (\leq) 0 \quad v^T \cdot \boldsymbol{M} \cdot v < 0 \;\; (v \neq 0)$$

their eigenvalues are smaller than (smaller than or equal to) zero

- **Indefinite Tensors**. Each tensor that is neither positive definite nor negative definite is indefinite.

Tensor Decompositions

- **Symmetric/Antisymmetric Part**. For non-symmetric tensors T, the decomposition into a symmetric part S and an antisymmetric part A is a common practice:

$$T = S + A$$

where $S = \frac{1}{2}(T + T^T)$, $A = \frac{1}{2}(T - T^T)$

$$\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

strain tensor!

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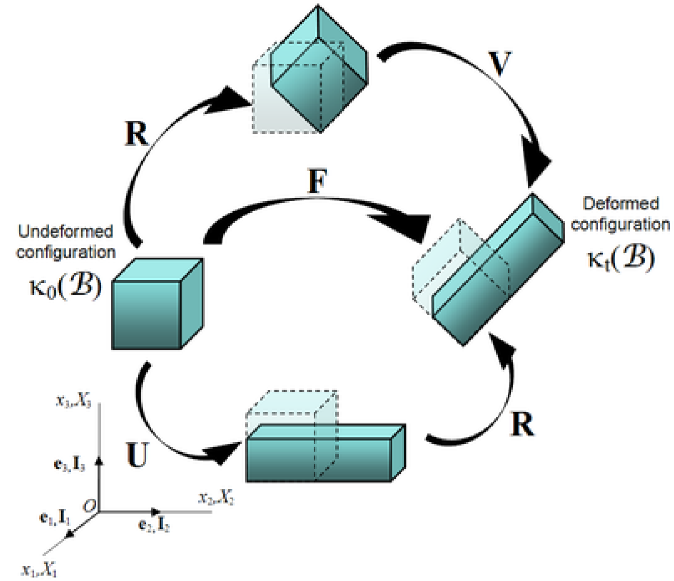
where $S = \frac{1}{2}(T + T^T)$, $A = \frac{1}{2}(T - T^T)$

- Physically, antisymmetric part contains rotational information and the symmetric part contains information about isotropic scaling and anisotropic shear

Tensor Decompositions

- **Stretch/Rotation**. Another useful decomposition of non-symmetric, positive-definite tensors T (e.g., deformation gradient tensors) is the polar decomposition. It decomposes the transformation represented by T in a two-stage process: a rotation R and a right stretch U or a left stretch V

$$T = R \cdot U = V \cdot R$$



- A tensor is called *stretch* if it is symmetric and positive definite. A tensor is called *rotation* if it is **orthogonal** with determinant equal to one.

$$\begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Tensor Decompositions

- **Shape/Orientation**. Via eigen-analysis symmetric tensors are separated into shape and orientation.
  - Here, **shape refers to the eigenvalues** and orientation to the eigenvectors.
  - Note that the **orientation** field is not a vector field due to the bi-directionality of **eigenvectors**

Tensor Decompositions

- **Isotropic/Anisotropic Part.** <u>Symmetric</u> tensors can be decomposed into an isotropic T_{iso} and an anisotropic (deviatoric) part D

$$T = \frac{1}{3} tr(T) I + (T - T_{iso})$$

Tensor Decompositions

- **Isotropic/Anisotropic Part.** Symmetric tensors can be decomposed into an isotropic T_{iso} and an anisotropic (deviatoric) part D

$$T = \frac{1}{3}tr(T)I + (T - T_{iso})$$

- From a physical point of view, the isotropic part represents a direction independent transformation (e.g., a uniform scaling or uniform compression); the deviatoric part represents the distortion (with volume preservation)

Asymmetric Tensor Decomposition

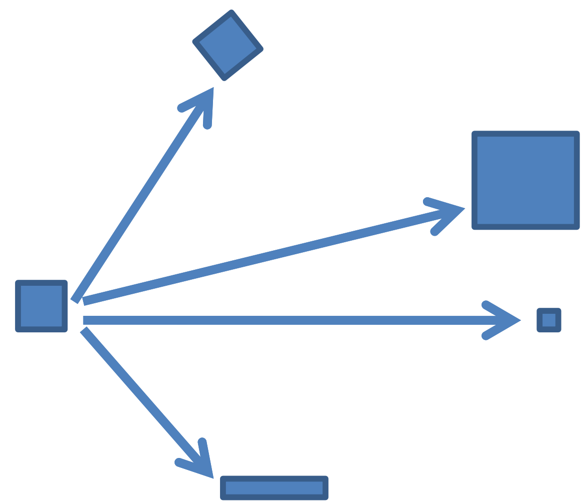
Jacobian $\begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \gamma_d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma_r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \gamma_s \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

– Isotropic scaling: $\gamma_d = \dfrac{a+d}{2}$

– Rotation: $\gamma_r = \dfrac{c-b}{2}$

– Anisotropic stretching: $\gamma_s = \dfrac{\sqrt{(a-d)^2 + (b+c)^2}}{2}$ $\quad \theta = \tan^{-1}\left(\dfrac{b+c}{a-d}\right)$



Second-order Tensor Field

Second-order Tensor Fields

- In visualization, usually not only a single tensor but a **whole tensor field** is of interest.

- *It can be considered as a function which assigns a tensor at any given position in space*.

we consider only **second-order** tensor which can be represented in the form of matrices.

What are the Characteristics?

or what can be visualized…

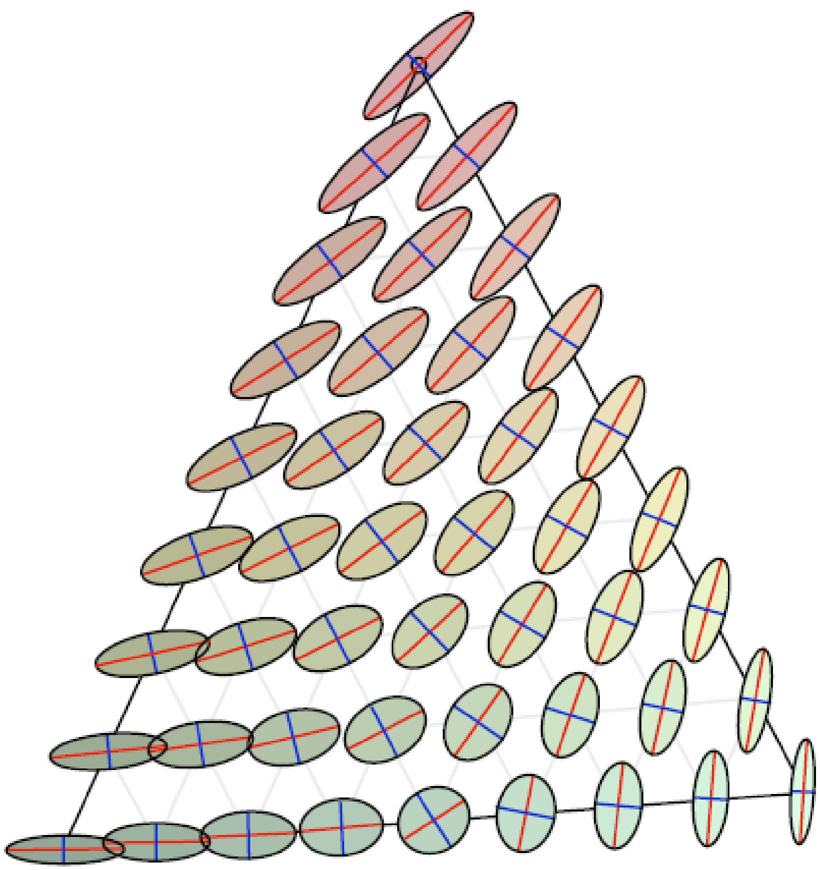
- **Scalar** related
  - Components (or individual entries)
  - Determinant
  - Trace
  - Eigen-values

- **Vector** related
  - Eigen-vectors

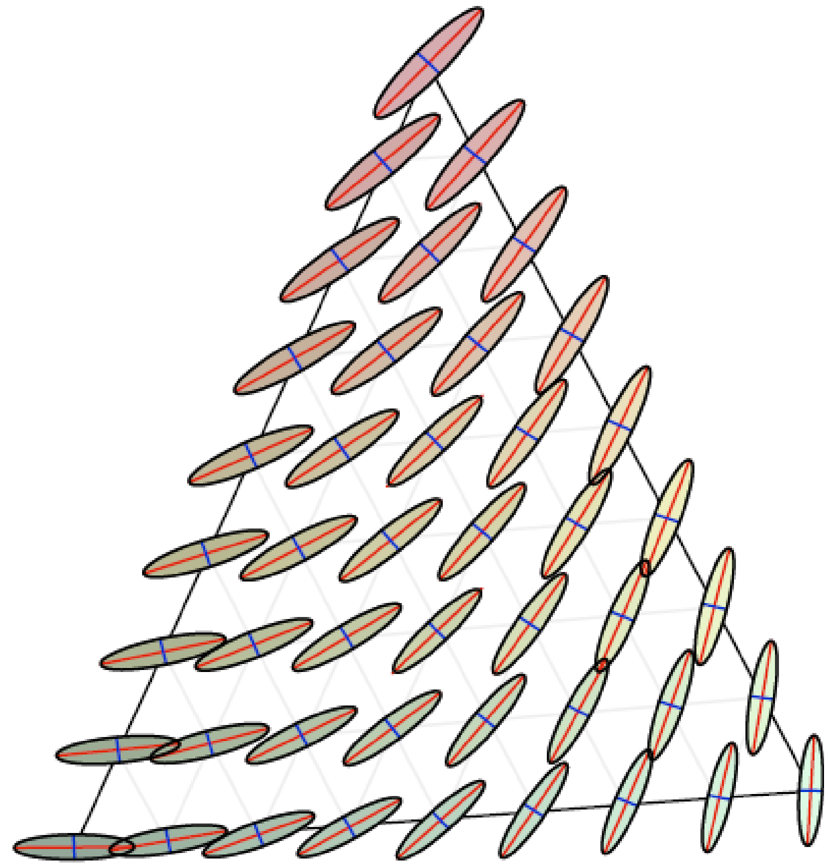
Tensor Interpolation

- Challenges
  - Natural representation of the original data.
    - This includes the **preservation of central tensor properties** (e.g., positive definiteness) and/or important scalar tensor invariants (e.g., the determinant).
  - Consistency.
    - consistent with the topology of the original data.
  - **Invariance**.
    - The resulting interpolation scheme needs to be invariant with respect to **orthogonal changes** of the reference frame.
  - Efficiency.
    - The challenge is to design an algorithm that represents a tradeoff between the criteria mentioned above and computational efficiency.

Tensor Interpolation



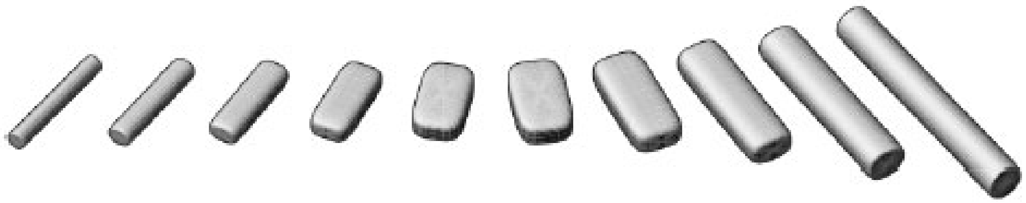
(a)



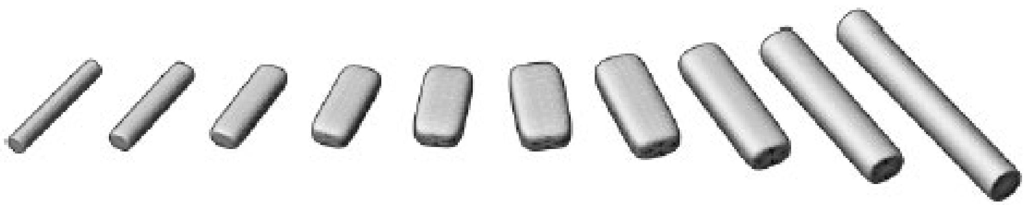
(b)

Comparison of component-wise tensor interpolation (a) and linear interpolation of eigenvectors and eigenvalues (b). Observing the tensors depicted by ellipses, the comparison reveals that the separate interpolation of direction and shape is much more shape-preserving (b).

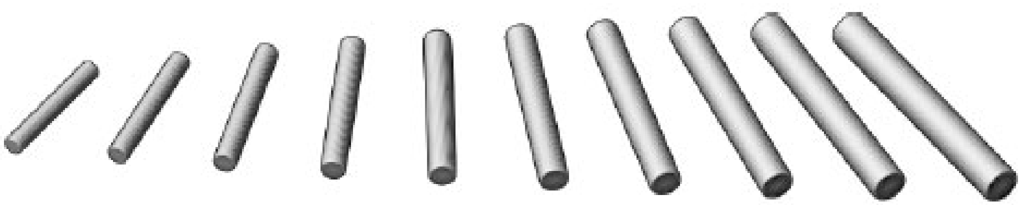
(a) Linear interpolation of tensor components: $(1-t)T_1 + tT_2$



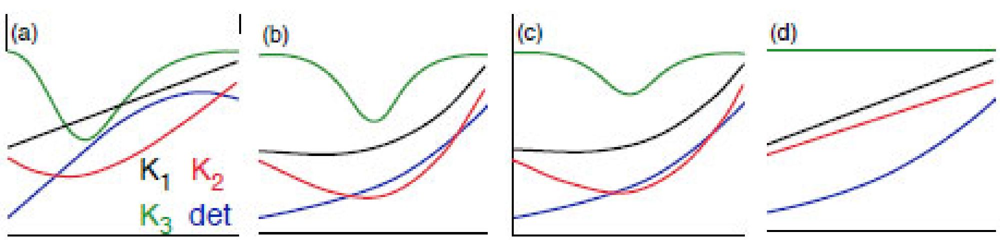
(b) Riemannian interpolation: $T_1^{1/2} \cdot (T_1^{-1/2} \cdot T_2 \cdot T_1^{-1/2})^t \cdot T_1^{1/2}$



(c) Log-Euclidean interpolation: $\exp((1-t)\log(T_1) + t\log(T_2))$



(d) Geodesic-loxodrome



(e)

Interpolations between two three-dimensional positive-definite tensors T1 and T2. The interpolation results are represented using superquadrics ((a)-(d)).

The plots (e) show the behavior of four tensor invariants for the respective interpolations: det(T), K1 =tr(T), K2 = ||D|| and K3 = det(D/||D||), where D is the deviator of T and ||.|| is the Frobenius norm. Image courtesy Kindlmann et al. 2007.

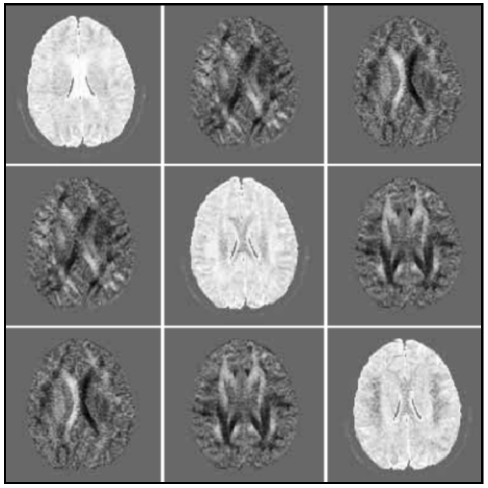
Challenges in Visualization

- **Hard to achieve intuitive visualization**
  - Tensors represent diverse quantities, ranging from the curvature of a surface to the diffusion of water molecules in tissue. *There is no universal intuition* similar to, e.g., arrows for vectors.
- **Multi-variate** nature makes it challenging.
  - The multi-variate nature of tensors affects all stages of the visualization pipeline, making each of them a challenging task, including interpolation, segmentation, and visualization.
- Perception issue: clutter and occlusion
- Highly application-dependent

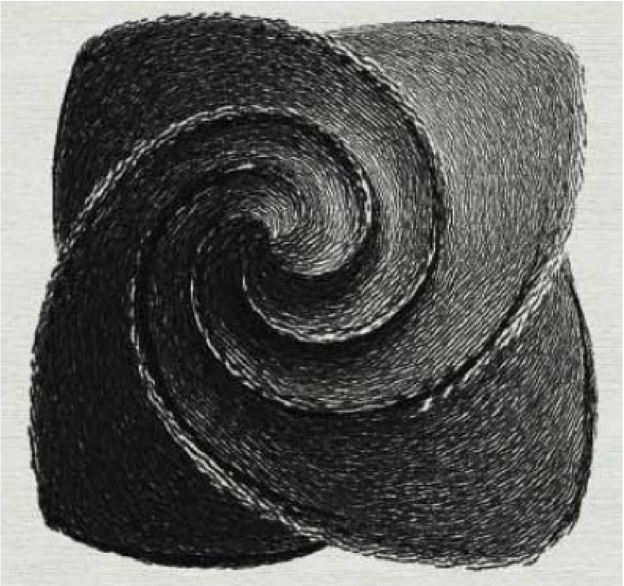
Overview of the Visualization Techniques for Tensors

Direct Methods

- Color plots or DVR for scalar properties

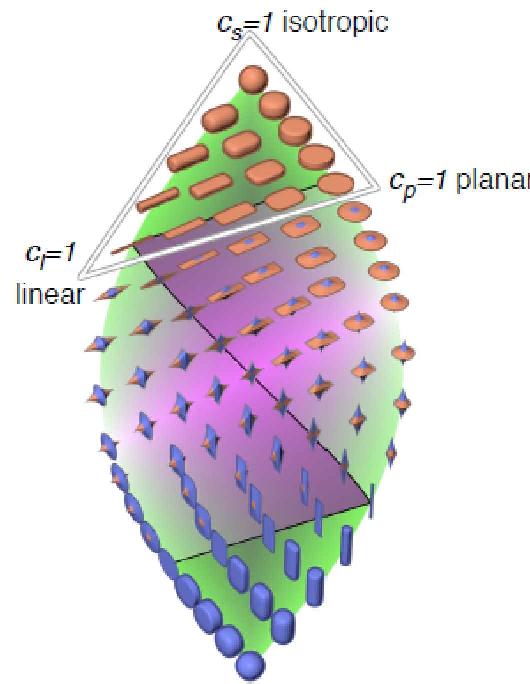


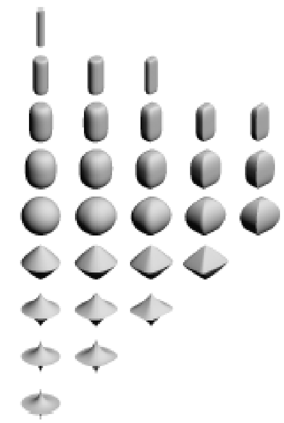
- Line field visualization for vector valued properties for s.p.d. tensor fields



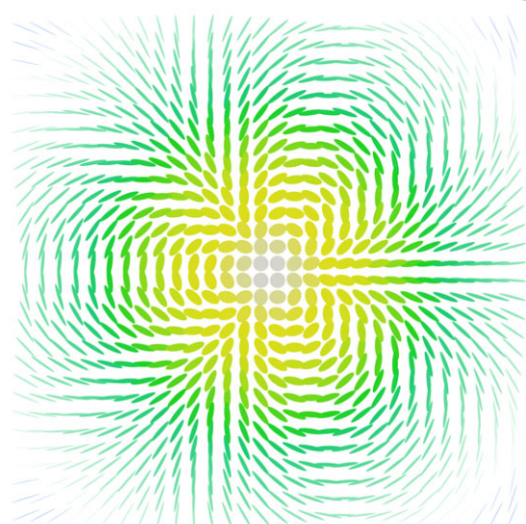
Direct Methods

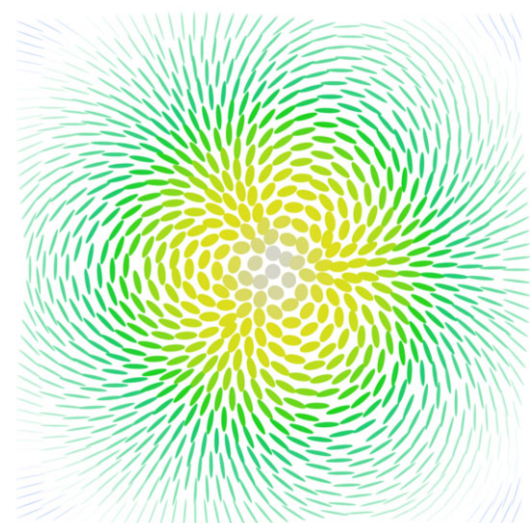
- Glyph-based for the visualization of the local properties of tensor
  - **Not really that direct!**





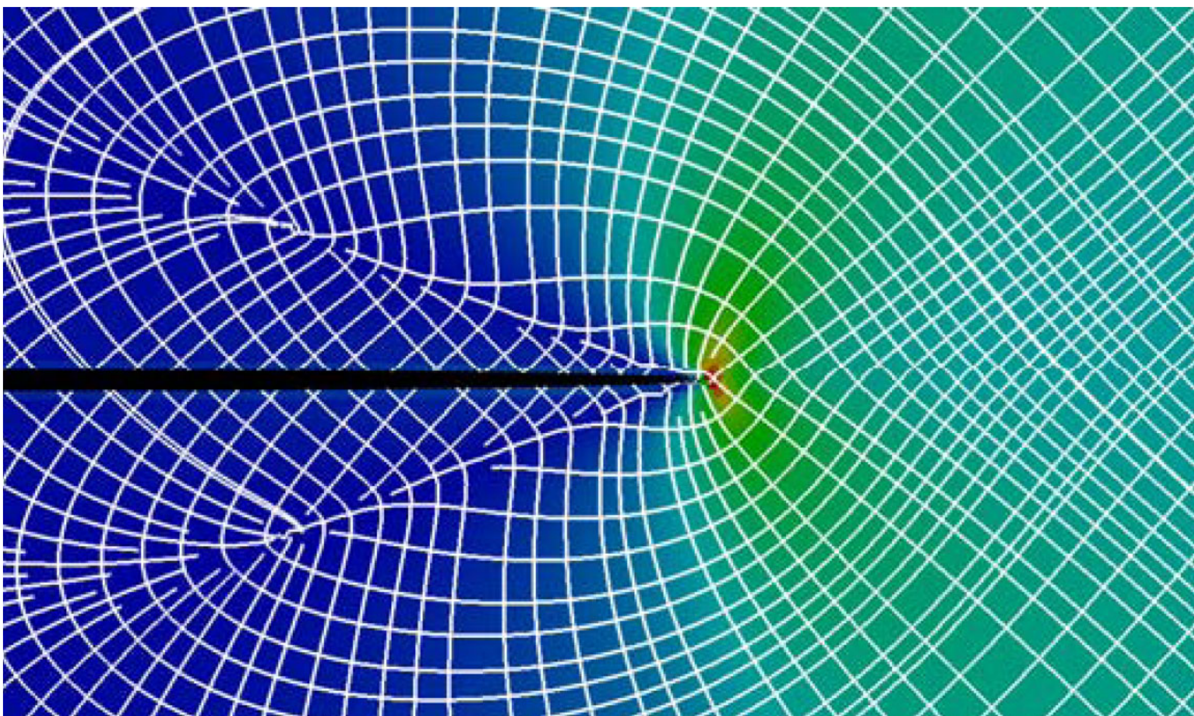
- Challenges
  - Glyph design
  - Glyph packing
    - where and how many

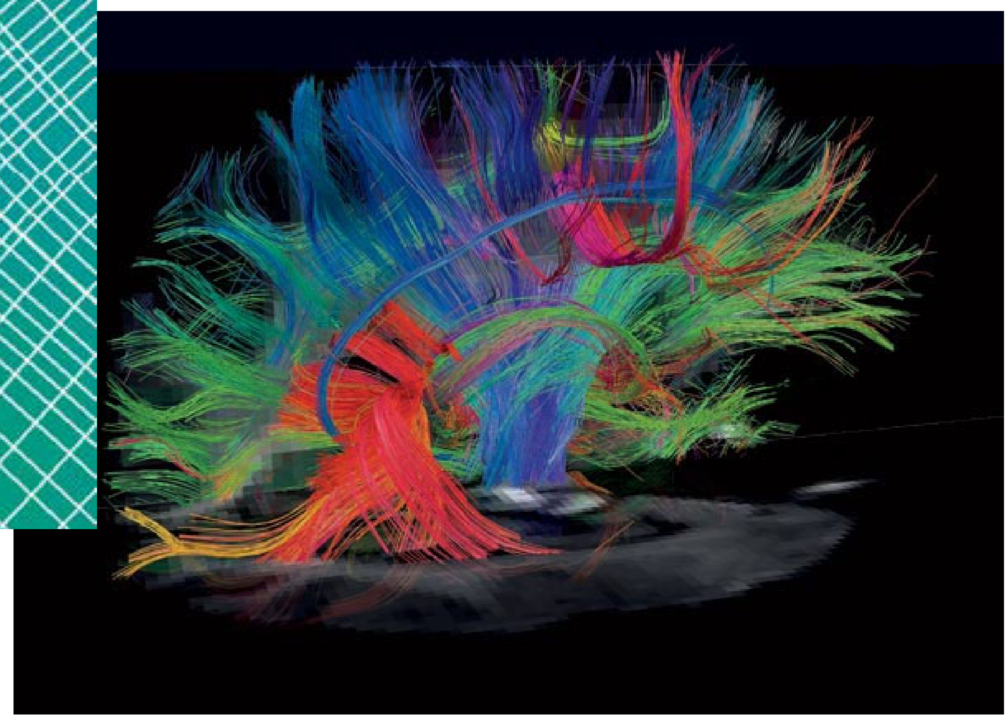




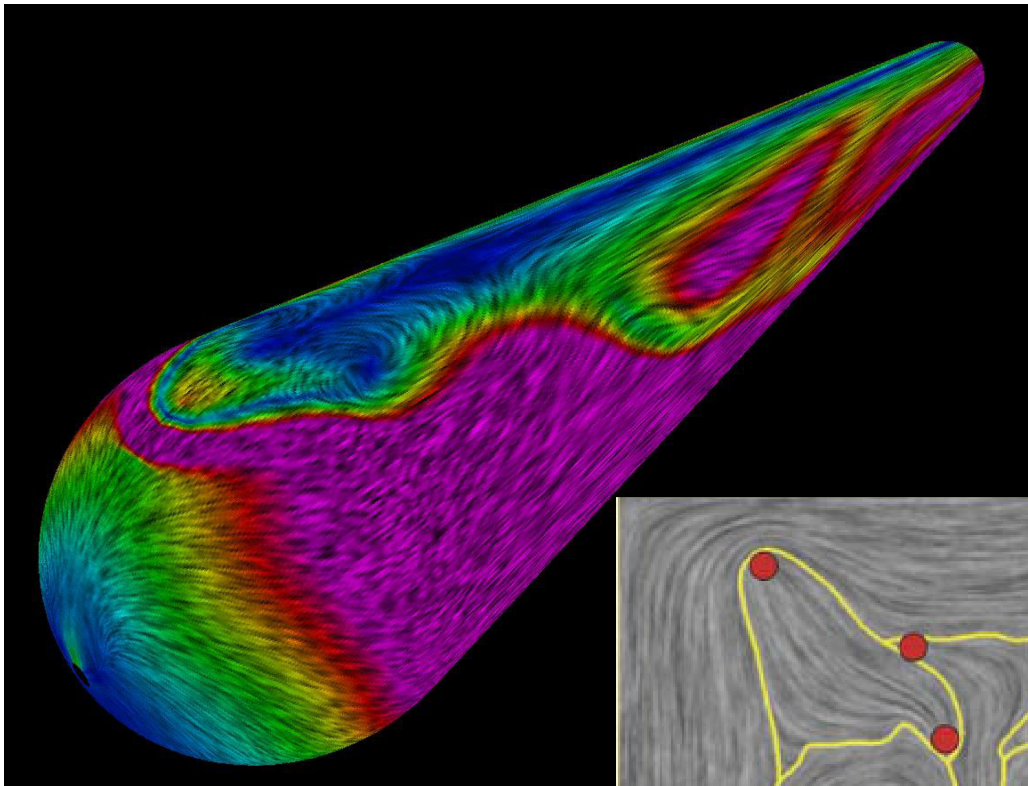
Geometric-Based Methods

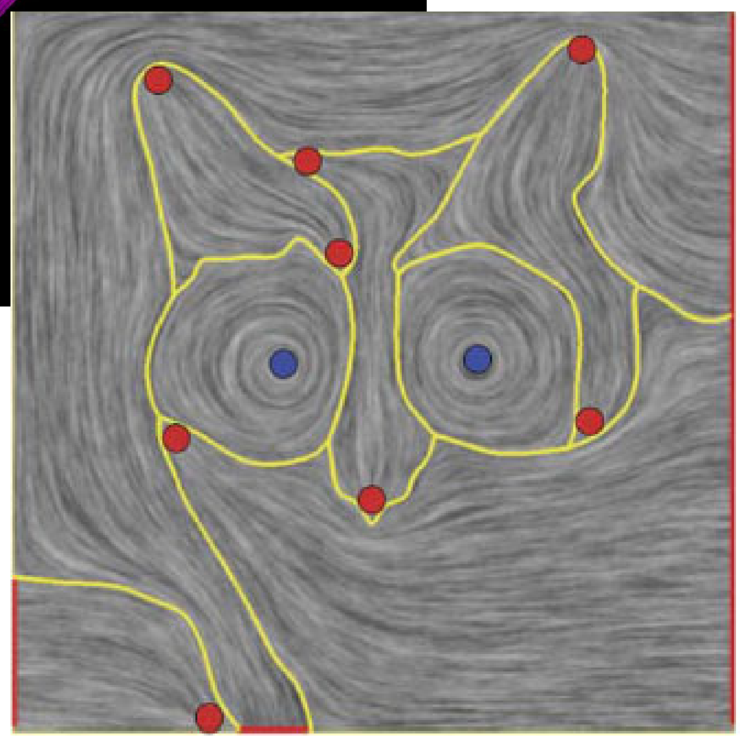
- Tensor lines and Hyper-streamlines
  - Integral curves of eigen-vector fields

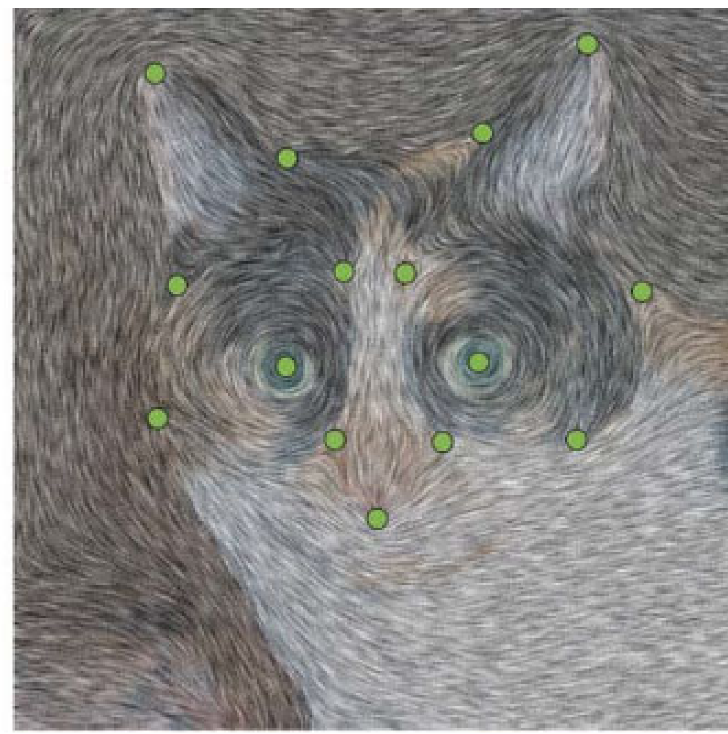




Texture-based Method

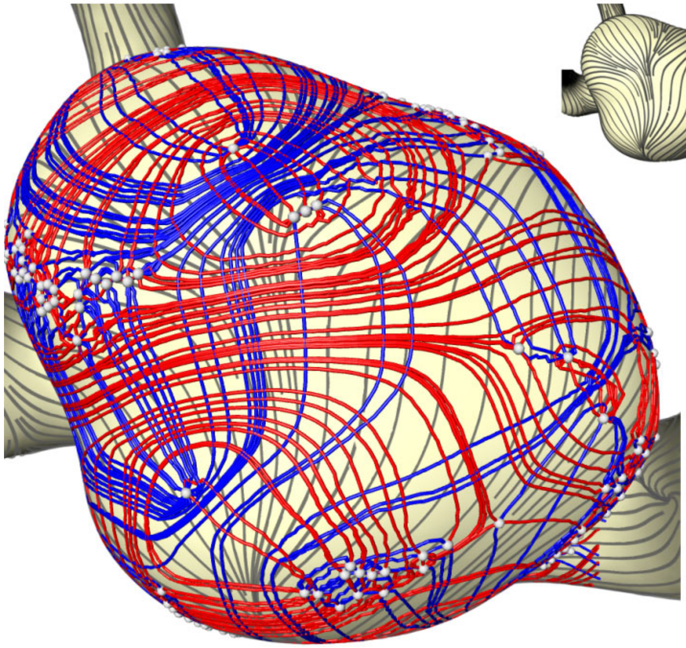




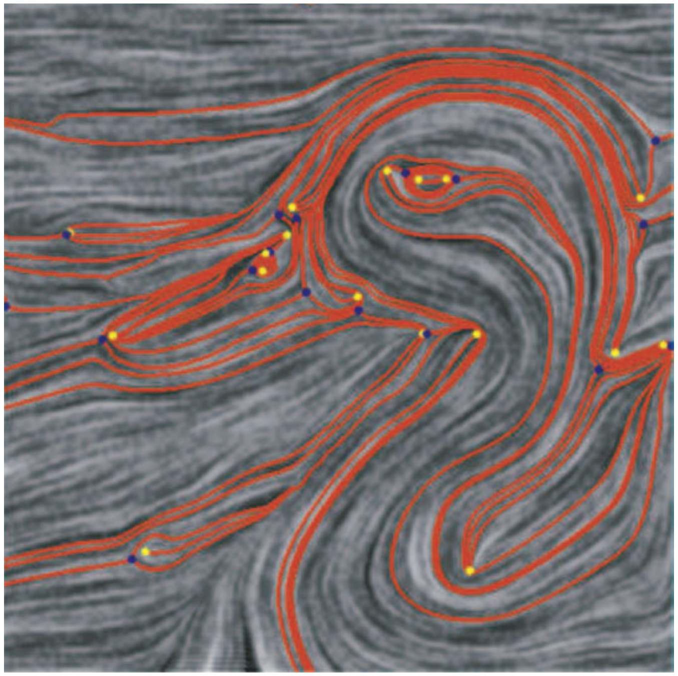


Feature-based Method

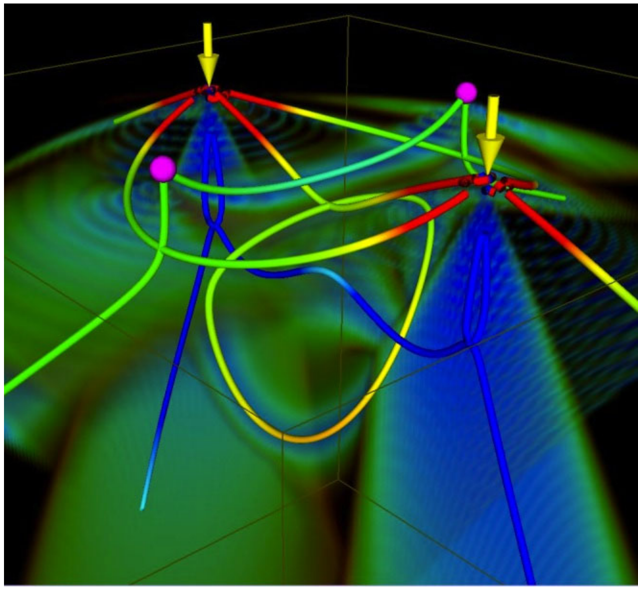
- Topology or other invariants



[Auer and Hotz EuroVis11]



[Zhang et al. TVCG 2007]



[Zheng and Pang, Vis04]