Morse-Smale Complex

Morse Theory

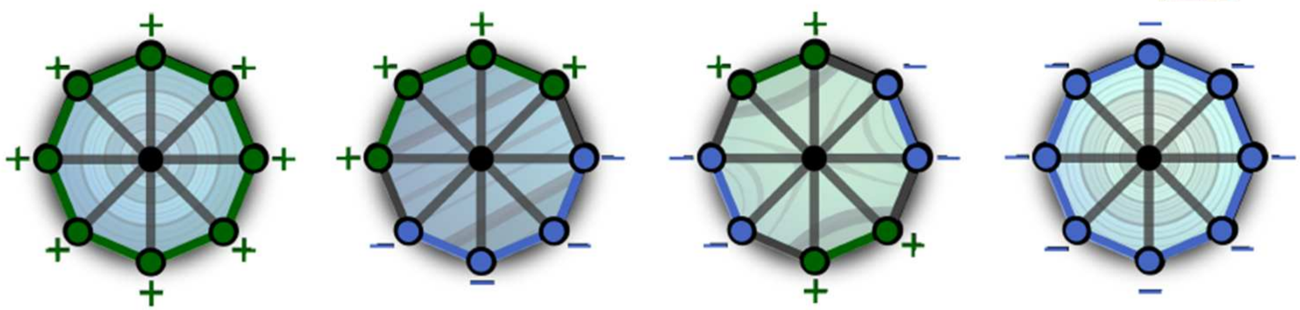
- Investigates the topology of a surface by looking at *critical points* of a function on that surface.

  $$\nabla f(p) = \left(\frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial y}(p) \right) = 0$$

- A function f is a Morse function if
  - f is smooth
  - All critical points are isolated
  - All critical points are non−degenerate $\det(Hessian(\boldsymbol{p})) \neq 0$

Notion of Critical Points and Their Index

- Minima, maxim, and saddles
- Topological changes
- Piecewise linear interpolation
- Barycentric coordinates on triangles
- Only exist at vertices



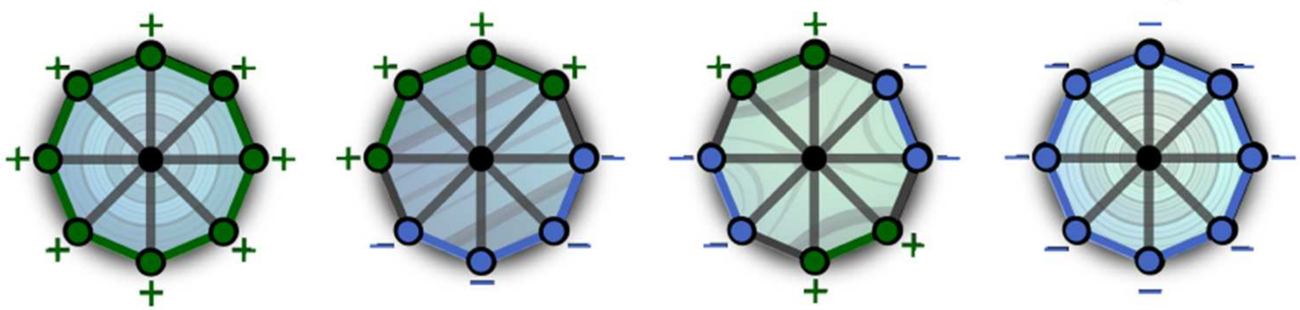
Minima Regular Saddles Maxima

Notion of Critical Points and Their Index

Standard form of a non-degenerate critical point p of a function $f: M^d \to R$

$$f = -{X_1}^2 - {X_2}^2 - \cdots - {X_\lambda}^2 + {X_{\lambda+1}}^2 + \cdots + {X_d}^2 + c$$

Where $(X_1, X_2, \cdots, X_n)$ are some local coordinate system such that p is the origin and $f(p) = c$.



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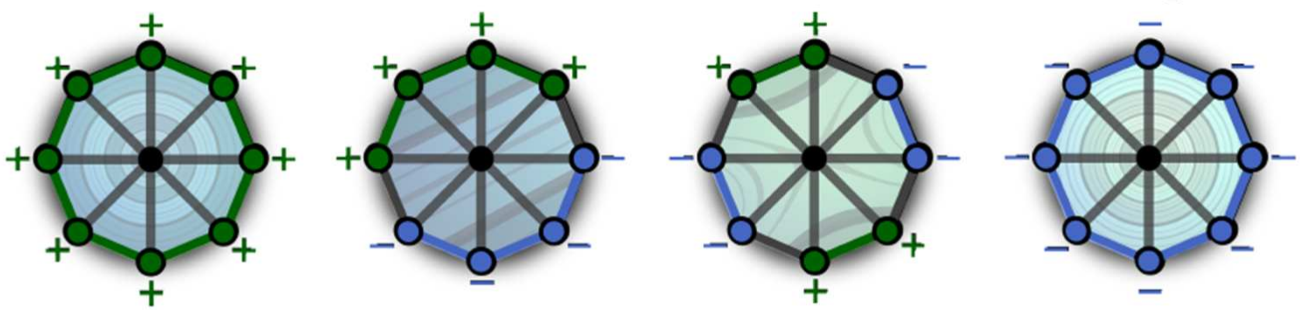
Where $(X_1, X_2, \cdots, X_n)$ are some local coordinate system such that p is the origin and $f(p) = c$.

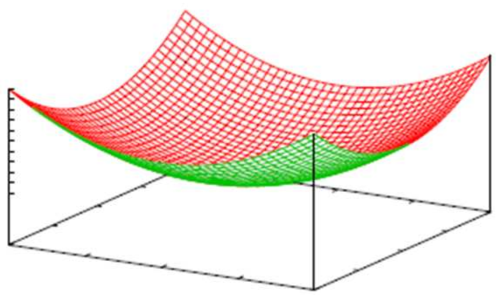
Then the number of minus signs, λ is the **index** of p.



Minima Regular Saddles Maxima

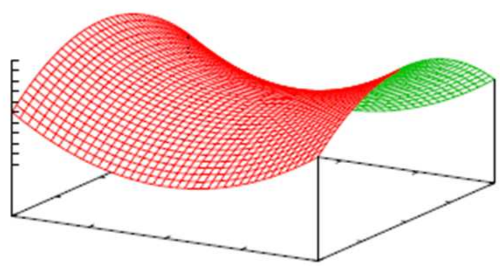
Notion of Critical Points and Their Index

Examples of critical points in 2-manfold



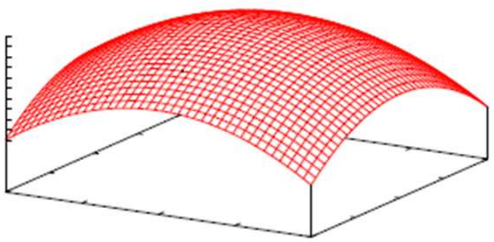
Minima

$x^2 + y^2$



Saddle

$x^2 - y^2$

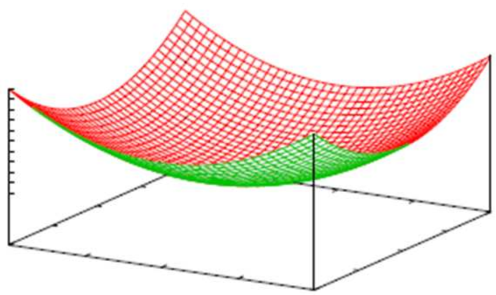


Maxima

$-x^2 - y^2$

Notion of Critical Points and Their Index

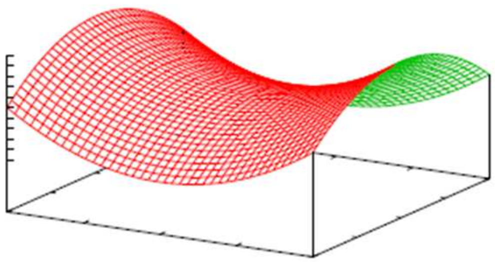
Examples of critical points in 2-manfold



Minima

$x^2 + y^2$

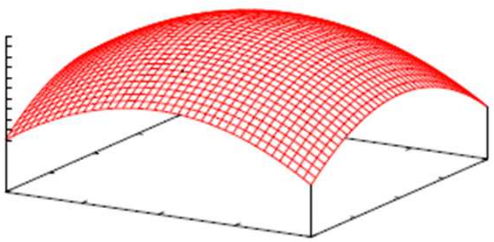
$\lambda = 0$



Saddle

$x^2 - y^2$

$\lambda = 1$



Maxima

$-x^2 - y^2$

$\lambda = 2$

Notion of Critical Points and Their Index

Standard form of a non-degenerate critical point p of a function $f: M^d \rightarrow R$

$$f = -{X_1}^2 - {X_2}^2 - \cdots - {X_\lambda}^2 + {X_{\lambda+1}}^2 + \cdots + {X_d}^2 + c$$

Where $(X_1, X_2, \cdots, X_n)$ are some local coordinate system such that p is the origin and $f(p) = c$.

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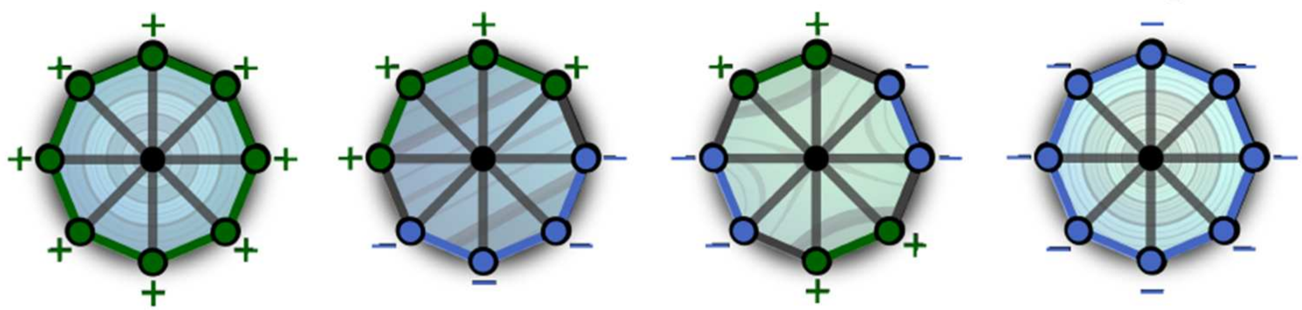


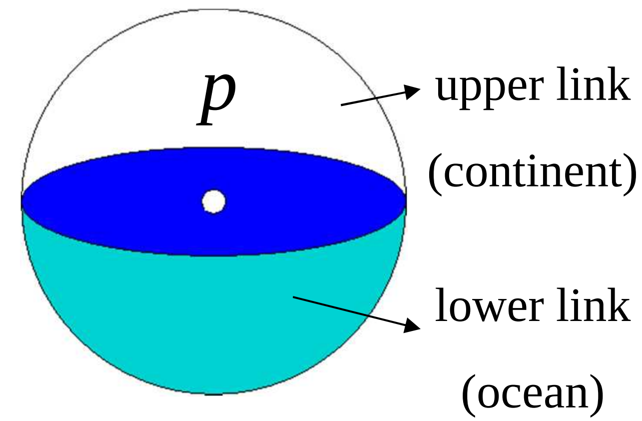
$\boldsymbol{i = 0}$ Regular $\boldsymbol{i = 1}$ $\boldsymbol{i = 2}$

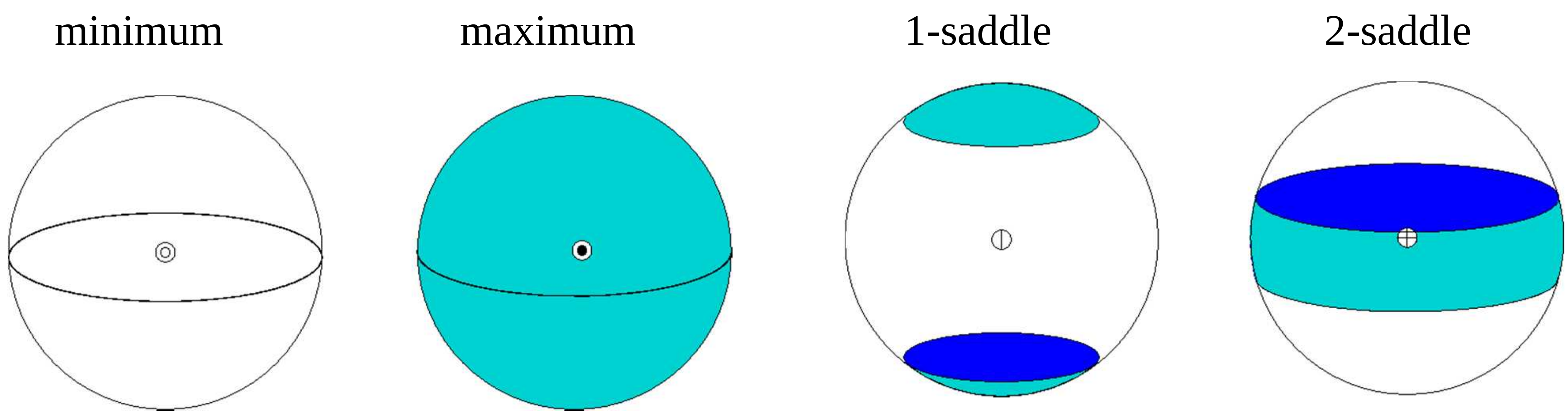
Critical Points in 3D

• Have zero gradient

• Characterized by lower link

regular





Critical Points in 3D

- Have zero gradient
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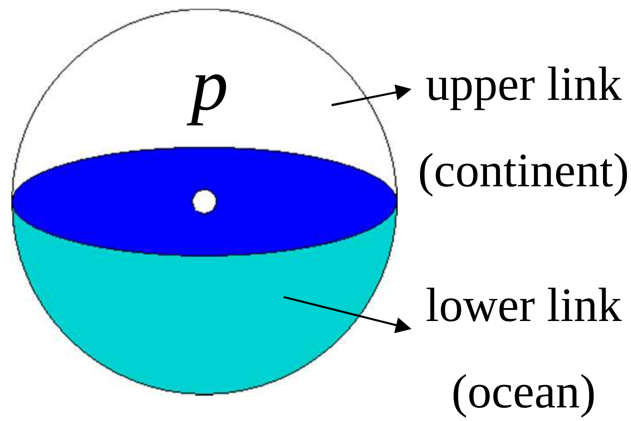
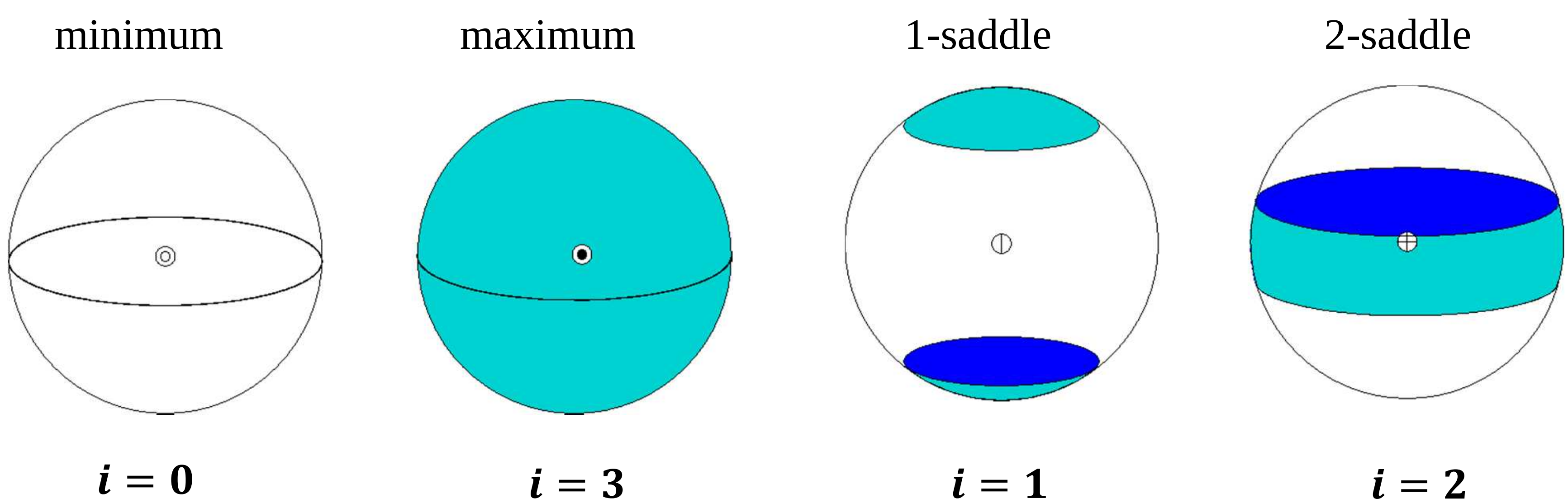


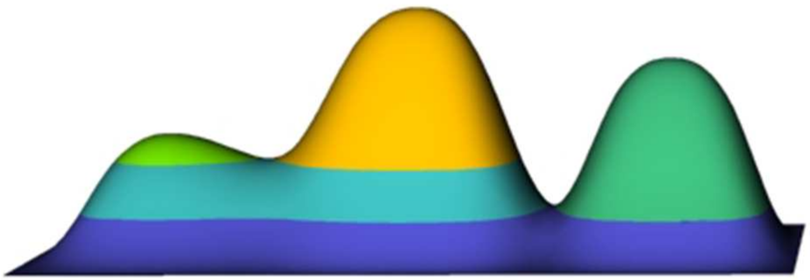


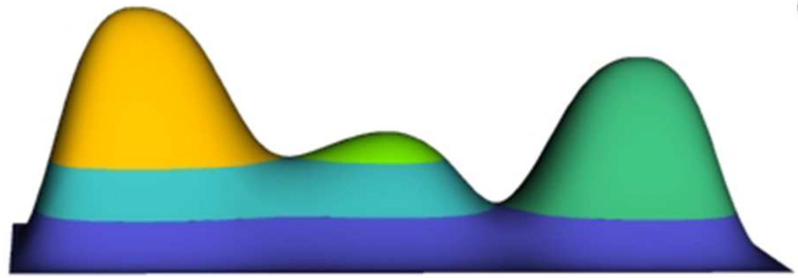
Reeb Graph

- The Reeb graph maps out the relationship between $index - 0$ and $index - 1$, and $index - (d - 1)$ and $index - d$ critical points in a d −dimensional space.
  – In 2-manifold, index(0) to index(1), and index(1) to index(2)
  – In 3-manifold, index(0) to index(1), and index(2) to index(3)

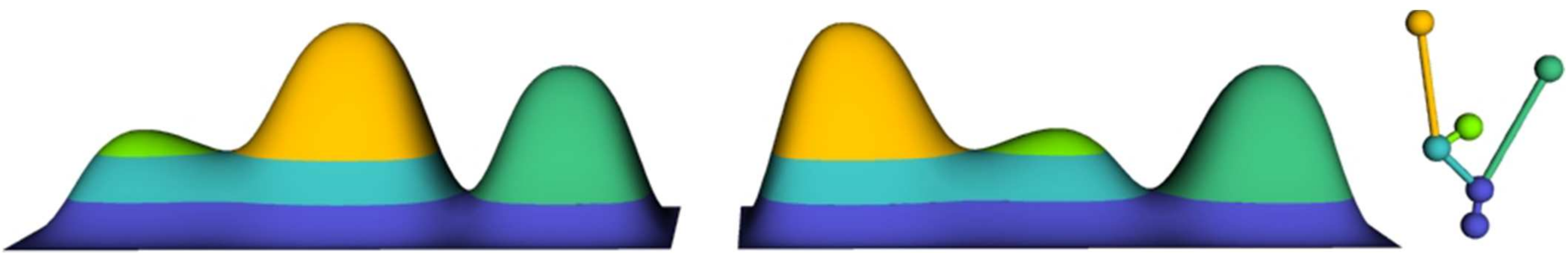
- The **contour tree** is a Reeb graph defined over a simply connected Euclidean space E^d

Limitation of Reeb Graph



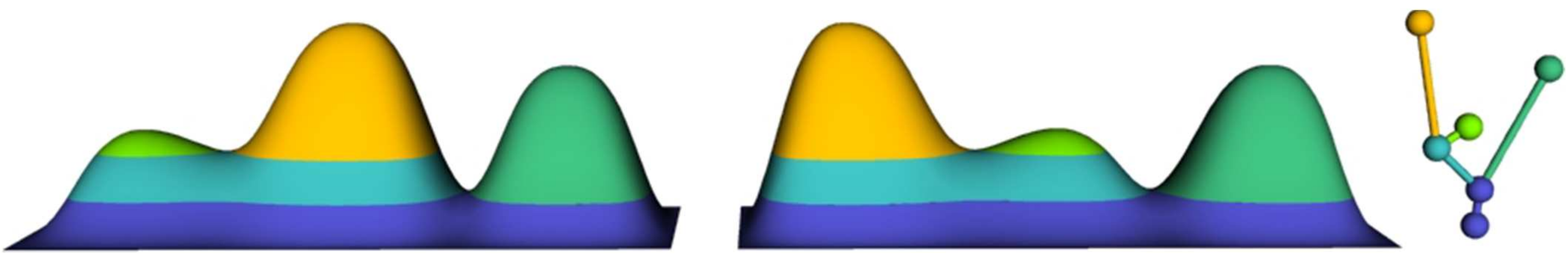


Limitation of Reeb Graph



Lacking the geometric connectivity of the features

Limitation of Reeb Graph



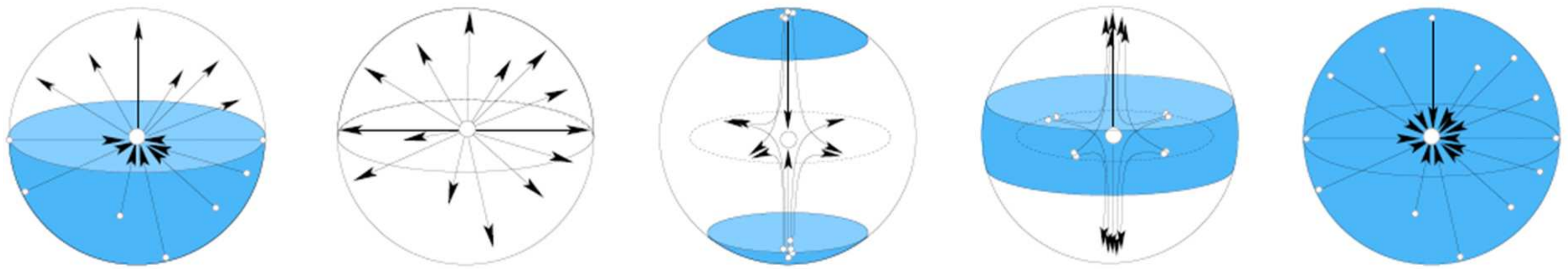
Lacking the geometric connectivity of the features

Additionally, for higher dimensional manifolds (>2), the saddle-saddle connections are not represented in the Reeb graph.

Morse Complex

- Instead of partitioning a manifold according to the behavior of level sets, it is more general to partition the manifold based on the behavior of the **gradient**.

- The gradient of a function defines a smooth **vector field** on M with zeroes at critical points.



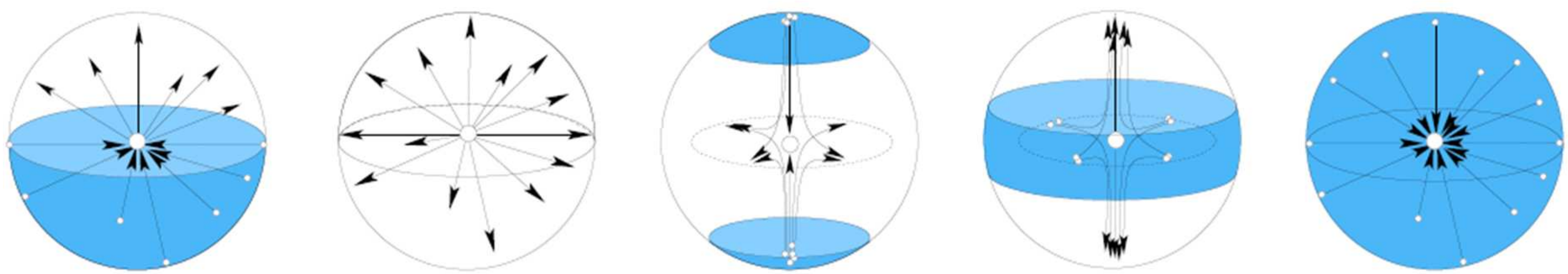
Morse Complex

- Integral line:

$$\frac{\partial}{\partial t} l(t) = \nabla f\big(l(t)\big) \qquad \text{for all } t \in R$$

  – Integral lines represent the flow along the gradient between critical points.

- Origin: $org(l) = lim_{t \to -\infty} l(t)$
- Destination: $dest(l) = lim_{t \to \infty} l(t)$



Morse Complex

- Integral lines have the following **properties**
  - Two integral lines are either disjoint or the same, i.e. uniqueness of each integral line
  - Integral lines cover all of M
  - The origin and destination of an integral line are critical points of f (except at boundary)
  - In gradient vector field, integral lines are monotonic, i.e. $org(l) \neq dest(l)$

Morse Complex

- Ascending/descending Manifolds
  - Let p be a critical point of $f: M \to R$.
  - The **ascending manifold** of p is the set of points belonging to integral lines whose origin is p.
  - The descending manifold of p is the set of points belonging to integral lines whose destination is p.

- Note that ascending and descending manifolds are also referred to as **unstable and stable** manifolds, **lower and upper disks**, and **right-hand and left-hand disks**.

Morse Complex

- Morse Complex
  - Let $f: M^d \rightarrow R$ be a Morse function. The complex of descending manifold of f is called the Morse complex

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- CW-complexes
  - Built on top of cells (0-cells, 1-cells, …., d-cells) via topologically gluing.
  - The *C* stands for "closure-finite", and the *W* for “weak topology".
  - Triangular mesh is one simple example of CW-complexes.

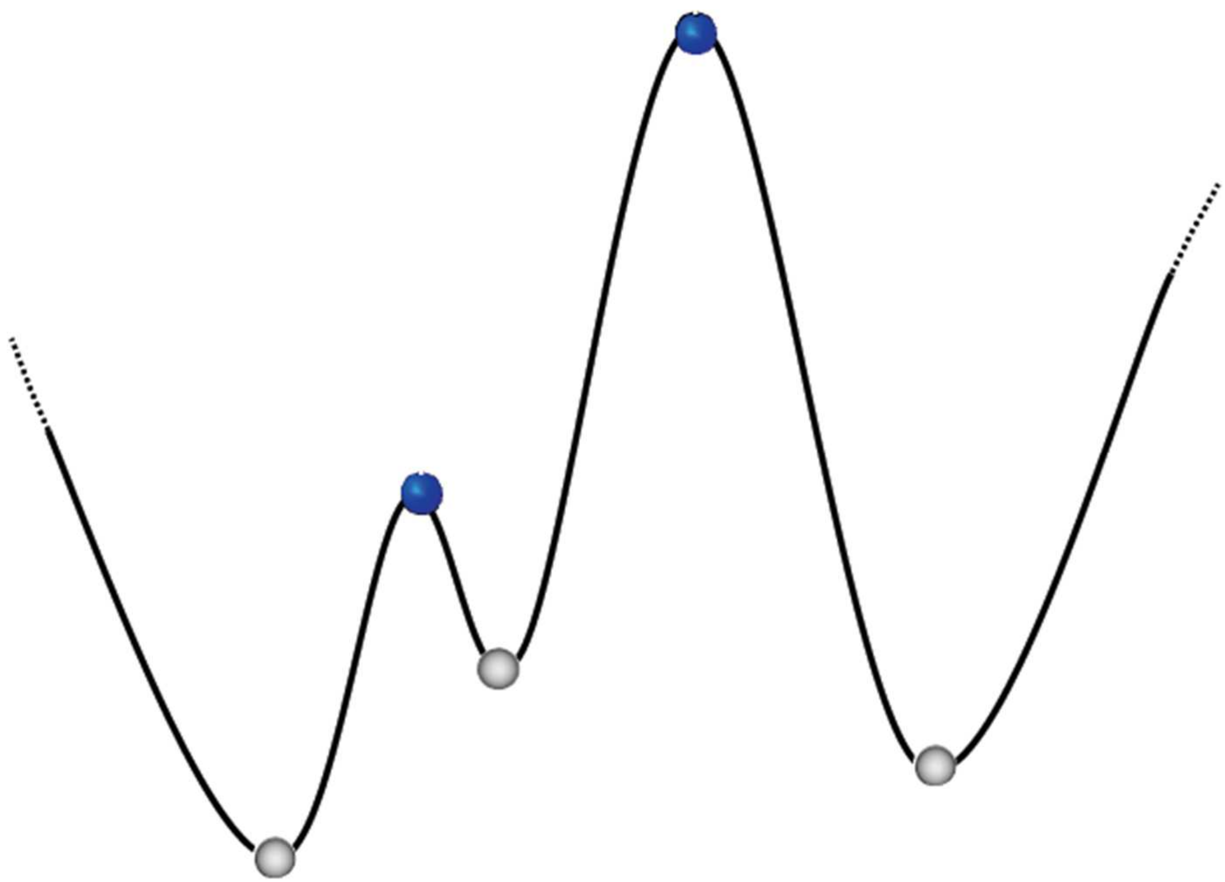
Morse-Smale Complex

- Morse-Smale Function
  - A Morse function f is Morse-Smale if the ascending and descending manifolds intersect only transversally.
    - Intuitively, an intersection of two manifolds as transversal when they are not “parallel” at their intersection.
  - A pair of critical points that are the origin and destination of a integral line in the Morse-Smale function cannot have the same index!
  - Furthermore, the index of the critical point at the origin is less than the index of the critical point at the destination.

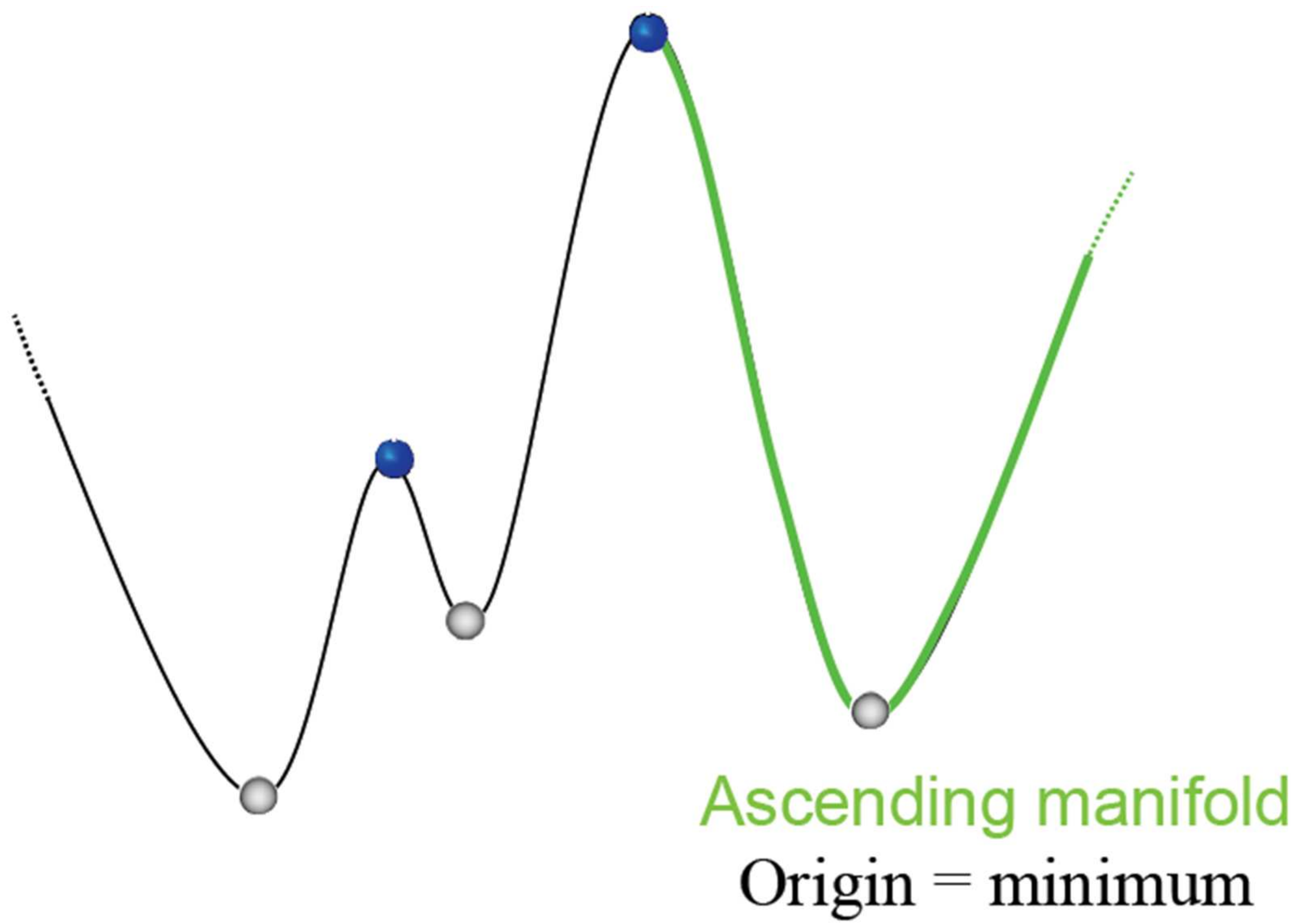
Morse-Smale Complex

- Given a Morse-Smale function f, the Morse-Smale complex of f is the complex formed by the intersection of the Morse complex of f and the Morse complex of $-f$.

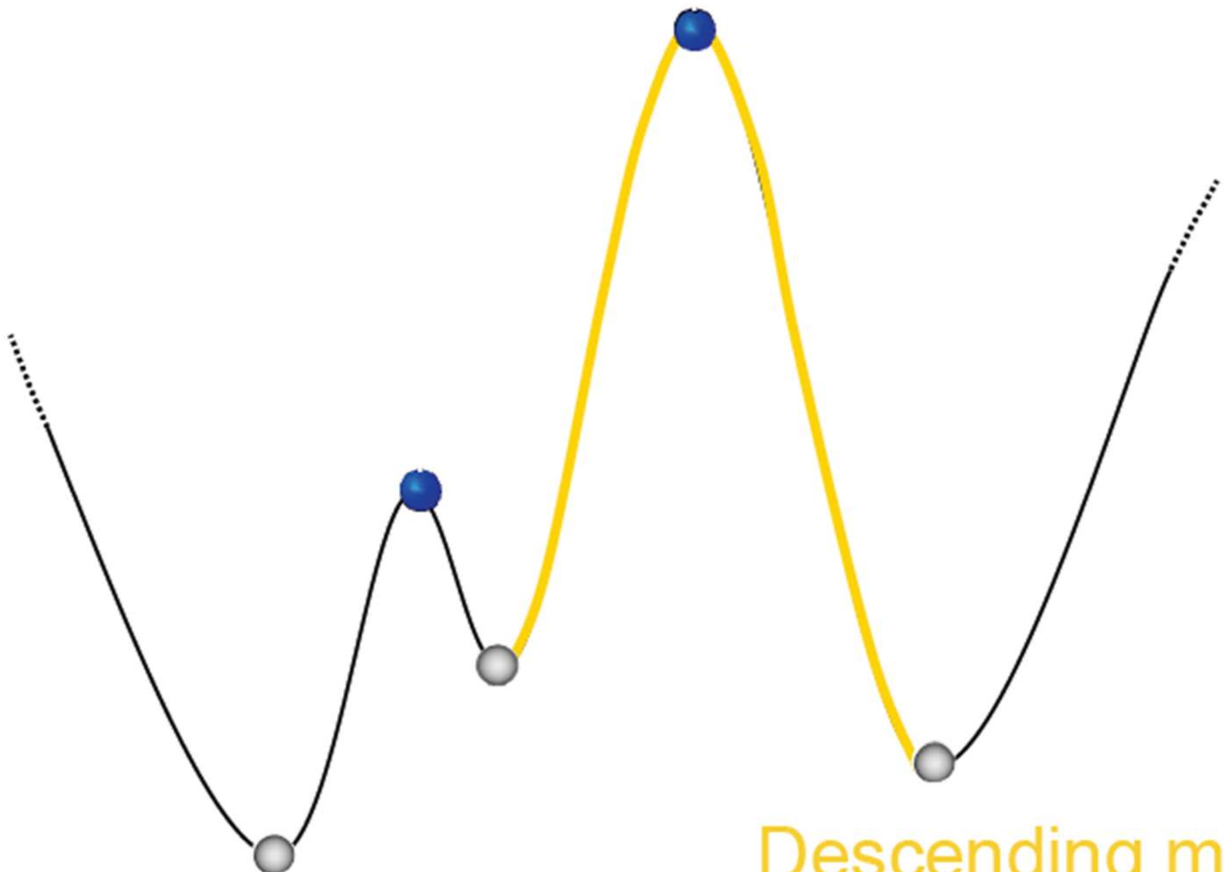
Morse-Smale Complex-1D



Morse-Smale Complex-1D



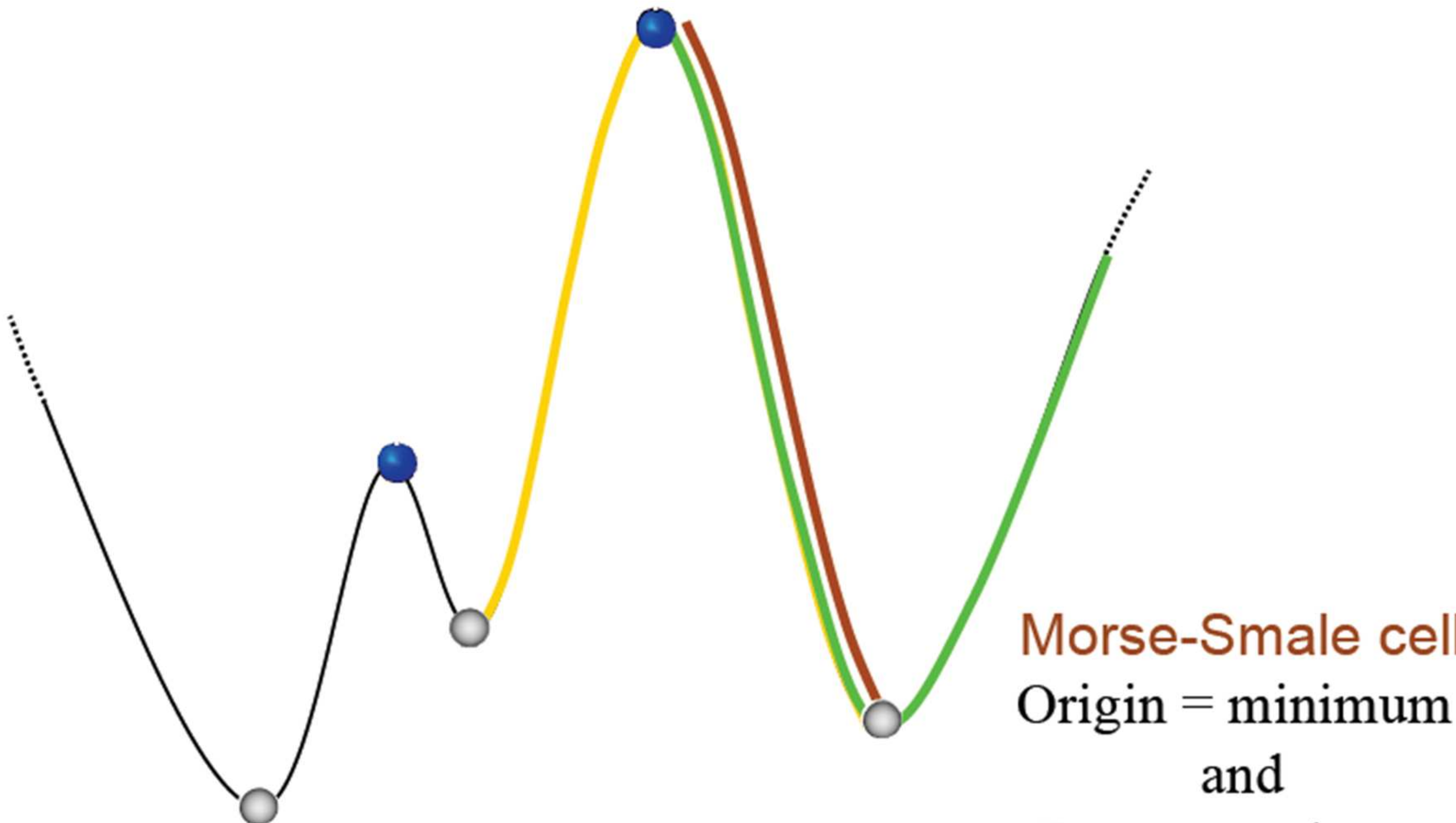
Morse-Smale Complex-1D



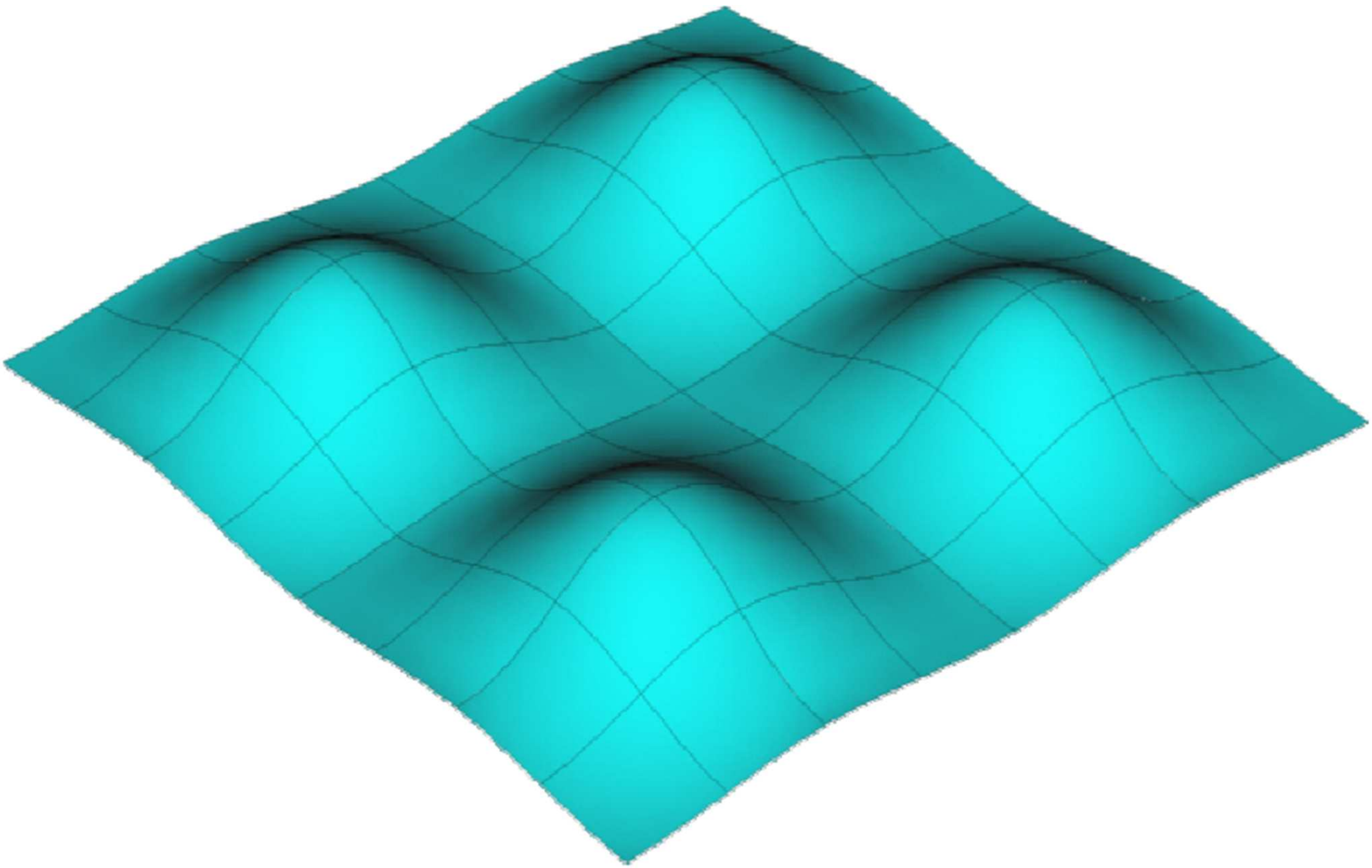
Descending manifold

Dest = maximum

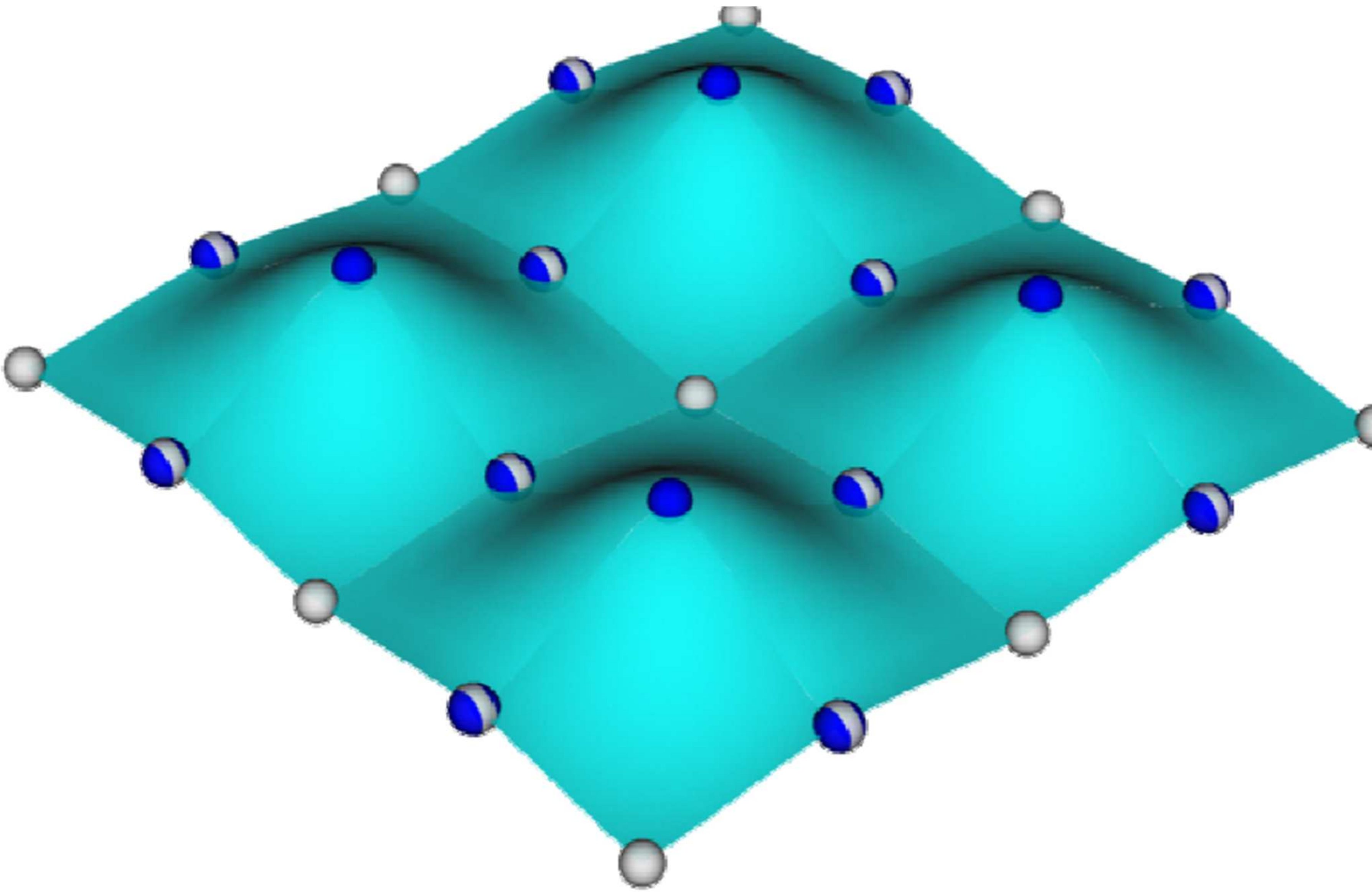
Morse-Smale Complex-1D



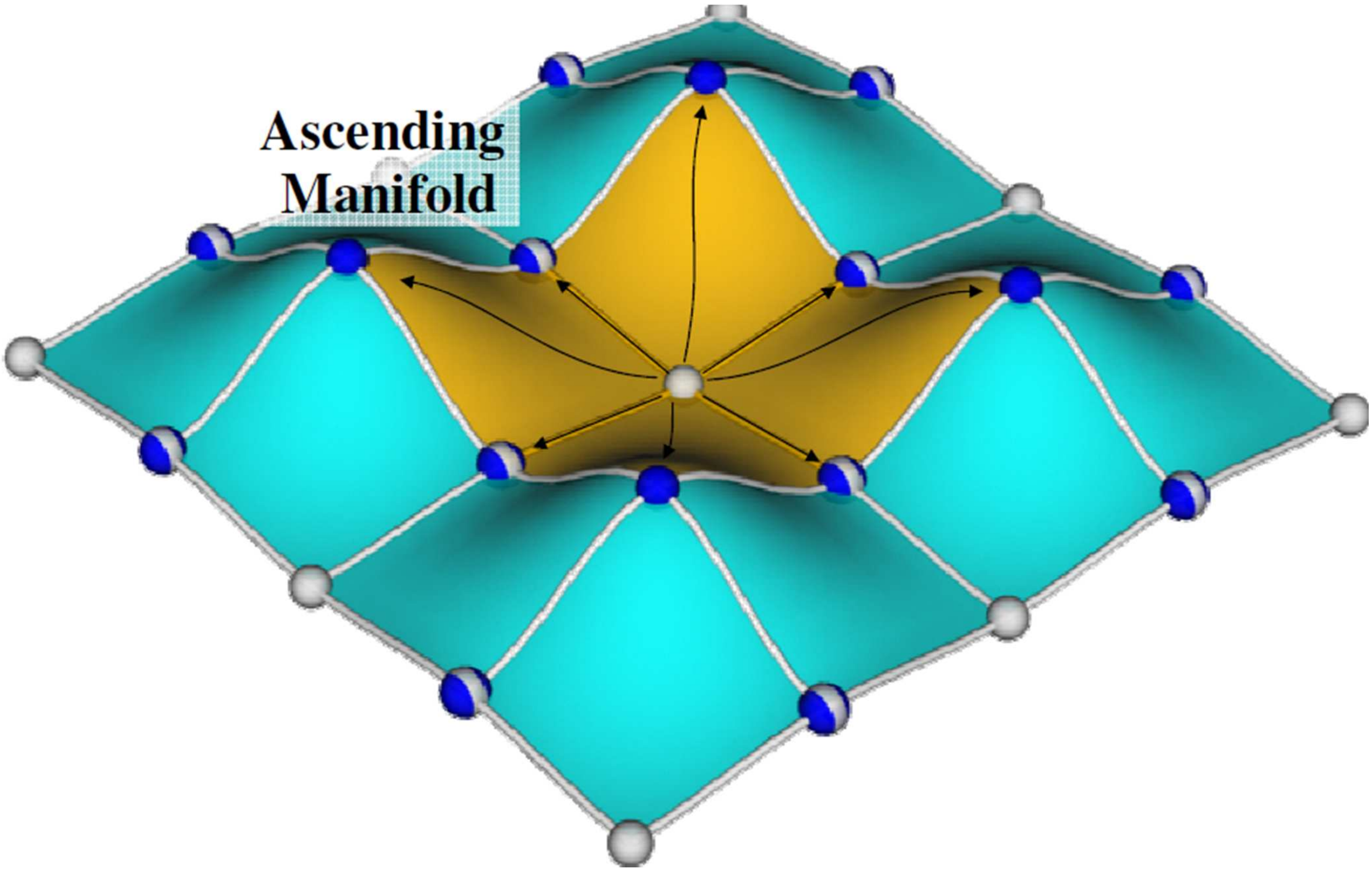
Morse-Smale Complex-2D



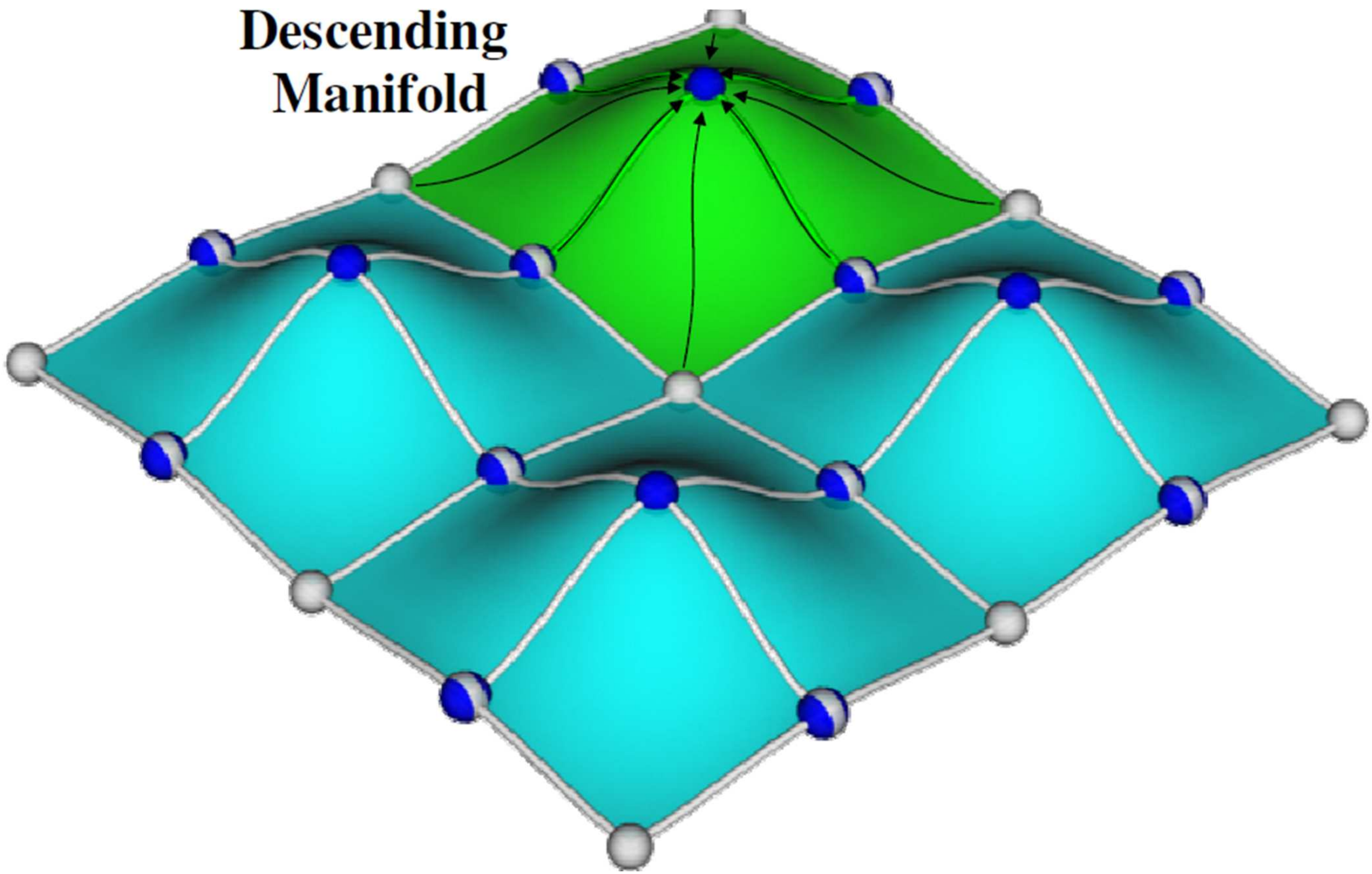
Morse-Smale Complex-2D



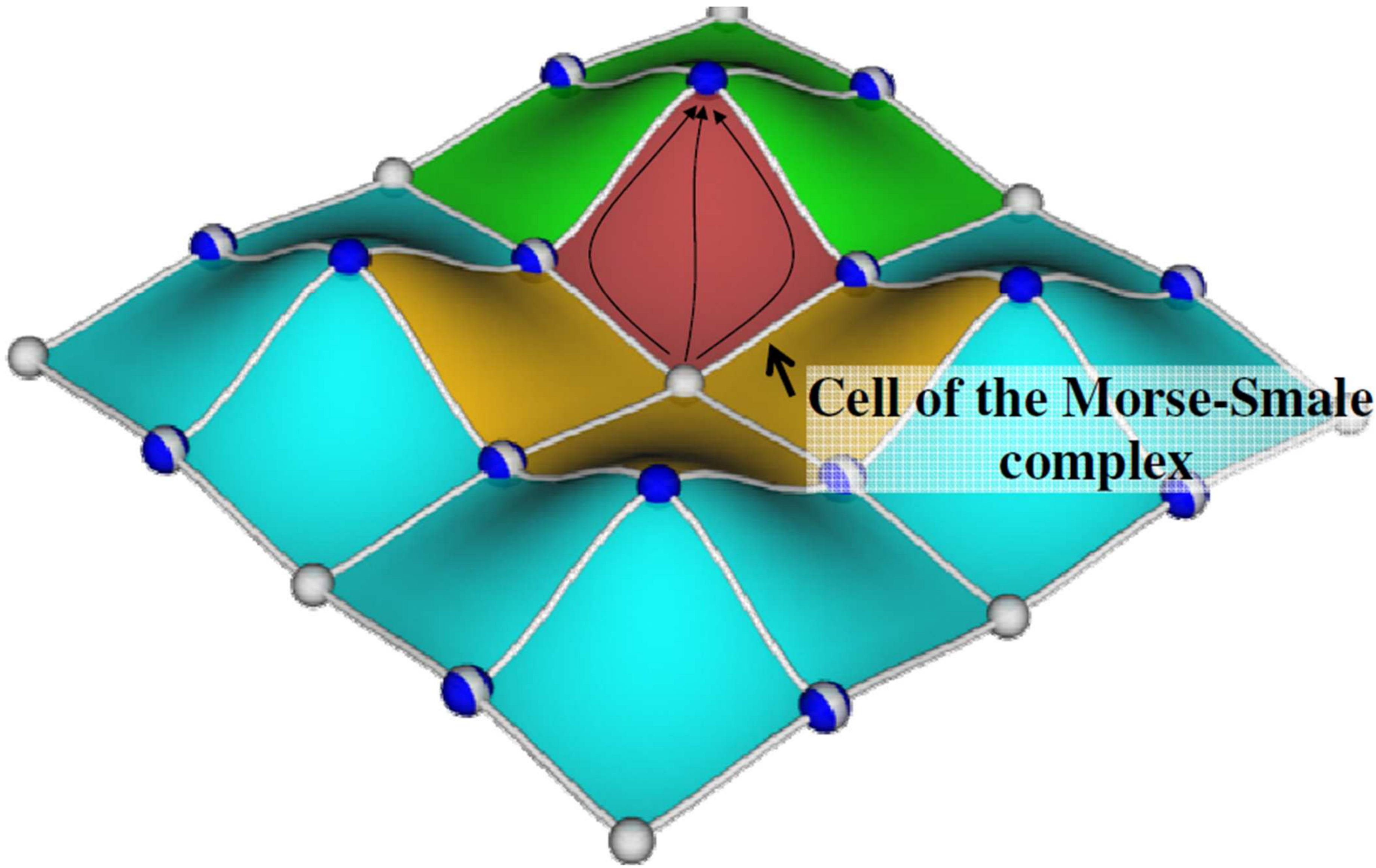
Morse-Smale Complex-2D



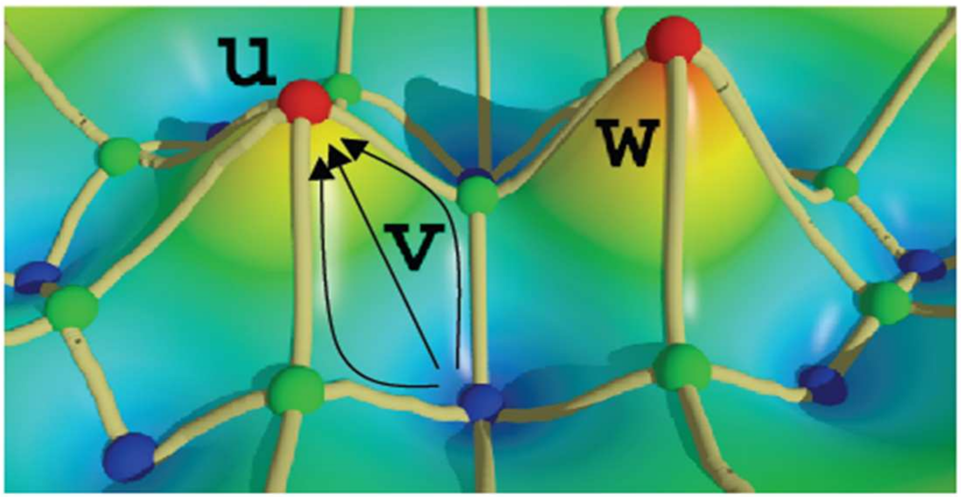
Morse-Smale Complex-2D

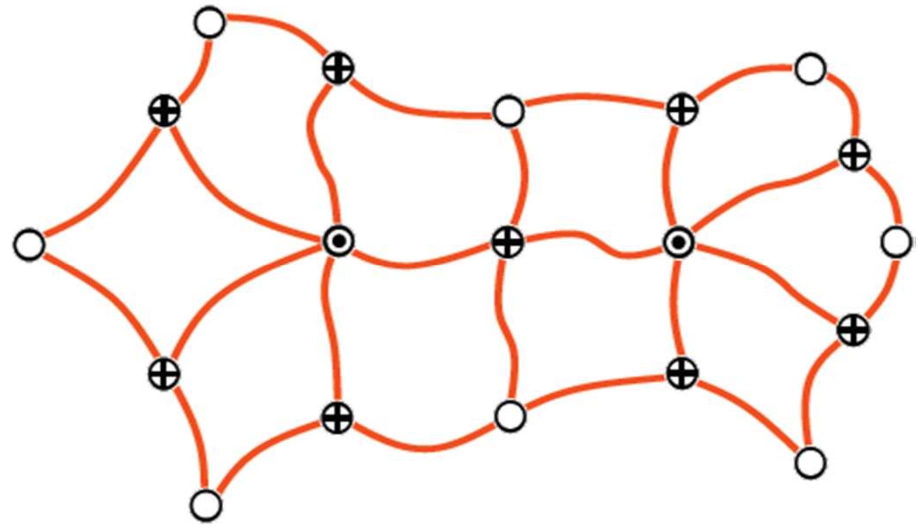


Morse-Smale Complex-2D



Morse-Smale Complex-2D

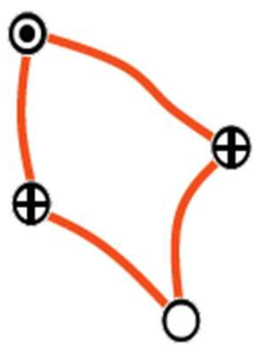




Decomposition into monotonic regions

Combinatorial Structure 2D

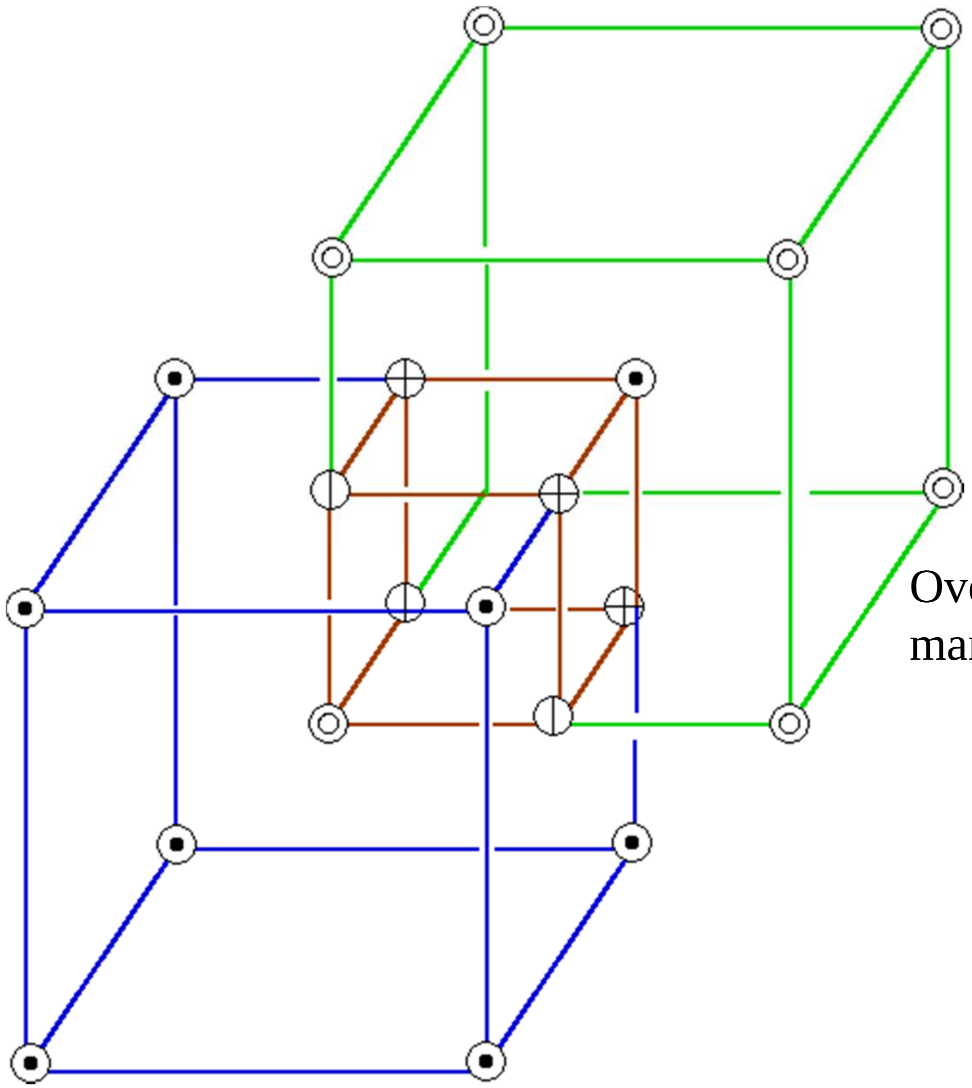
- Nodes of the MS complex are exactly the critical points of the Morse function
- Saddles have exactly four arcs incident on them



All regions are quads

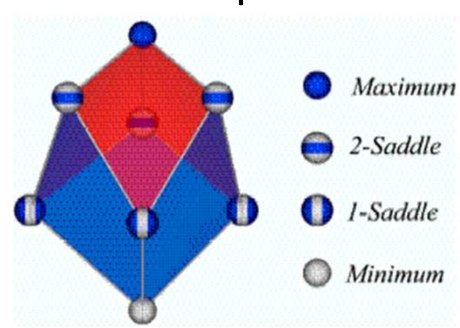
- Boundary of a region alternates between saddle-extremum
- 2k minima and maxima

Morse-Smale Complex in 3D



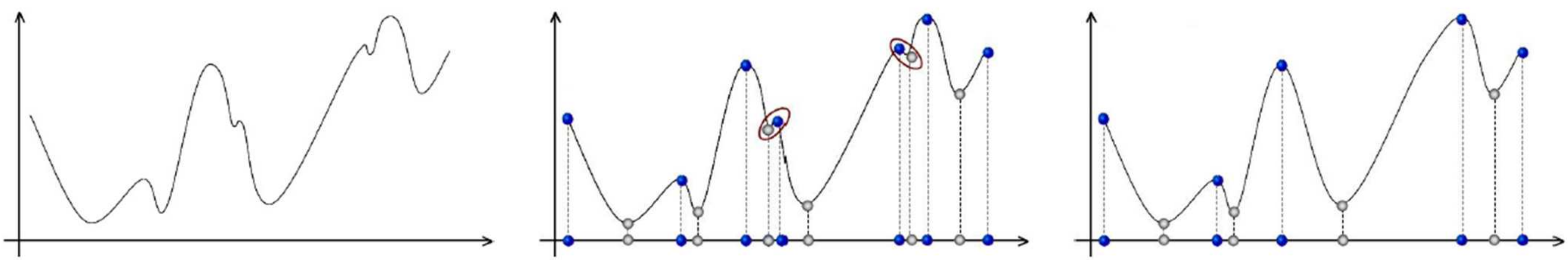
Overlay of Asc and Desc manifolds

3D MS Complex cell



Topological Simplification

(Persistence) Let p_a be the critical point creating a boundary component B and p_b the critical point destroying B, then the pair (p_a, p_b) is a persistence pair. The difference is function value $|f(p_a) - f(p_b)|$ is called the persistence of the topological feature (p_a, p_b)



(a) (b) (c)

Topological Simplification

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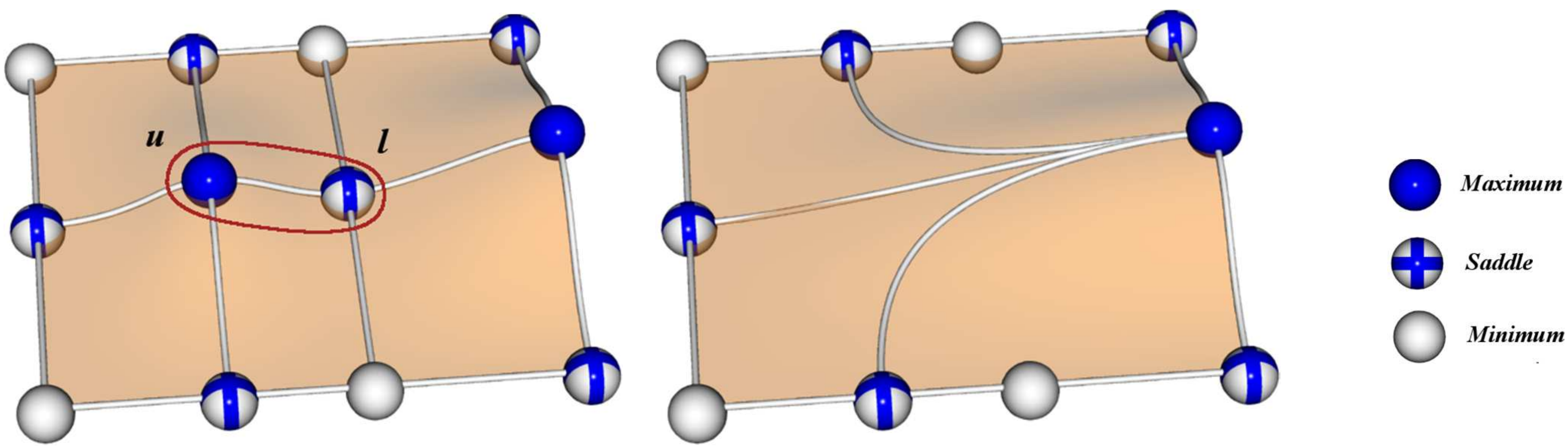
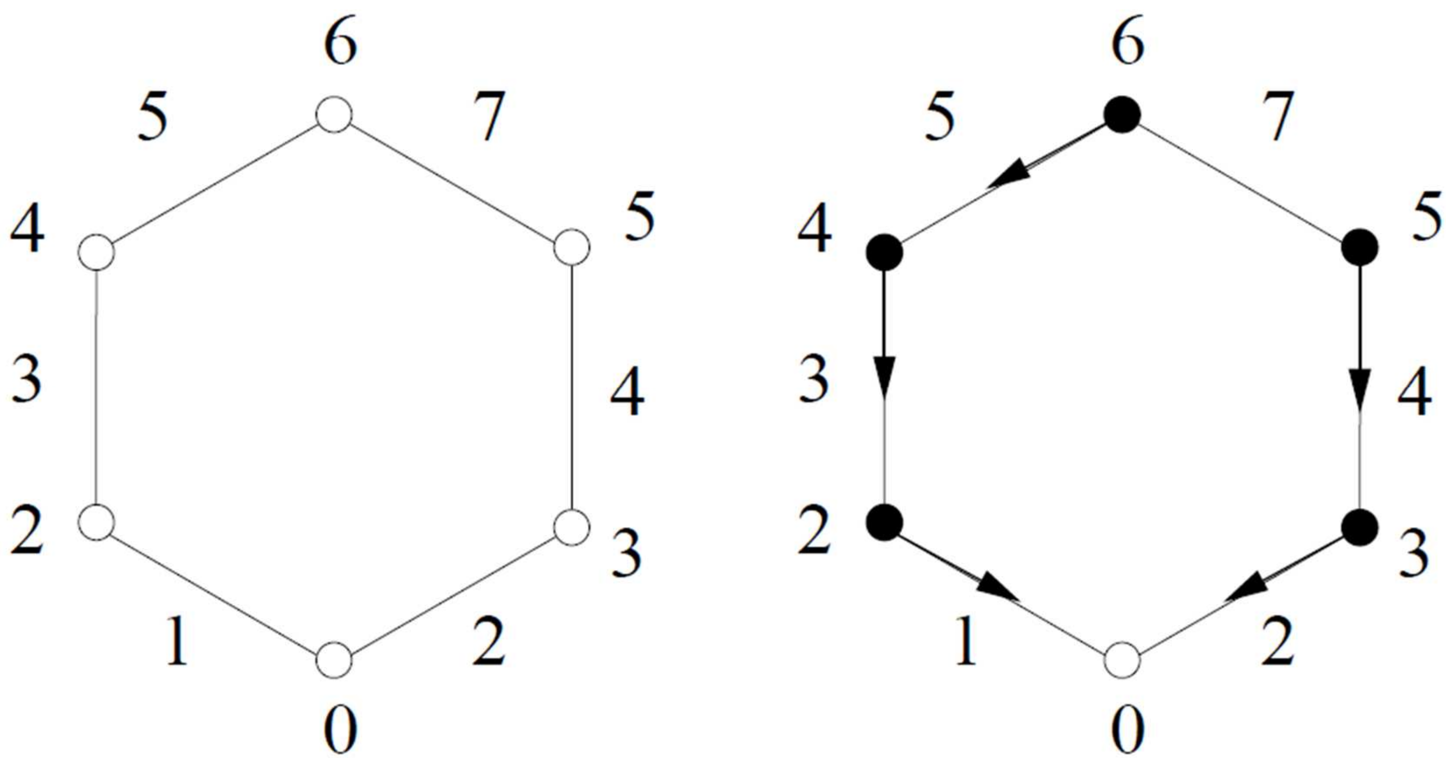


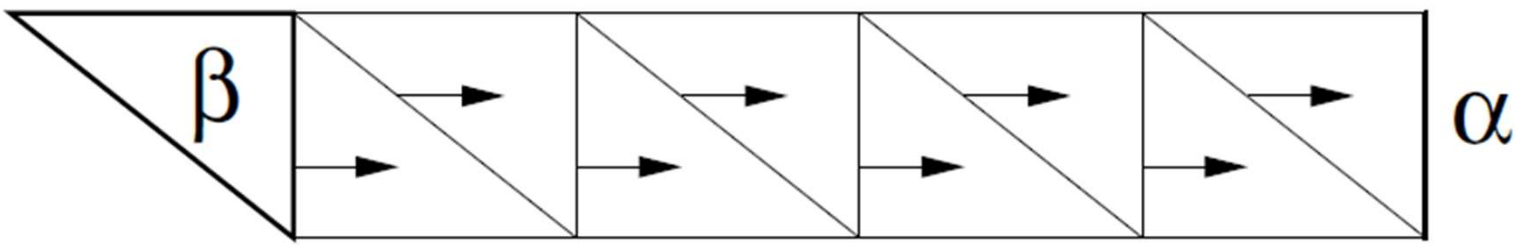
Discrete Morse-Smale Complex



The gradient directions (the arrows) are always pointing from lower-dimensional cells to their neighboring cells that are exactly one-dimension higher.

Discrete Morse-Smale Complex

V-path

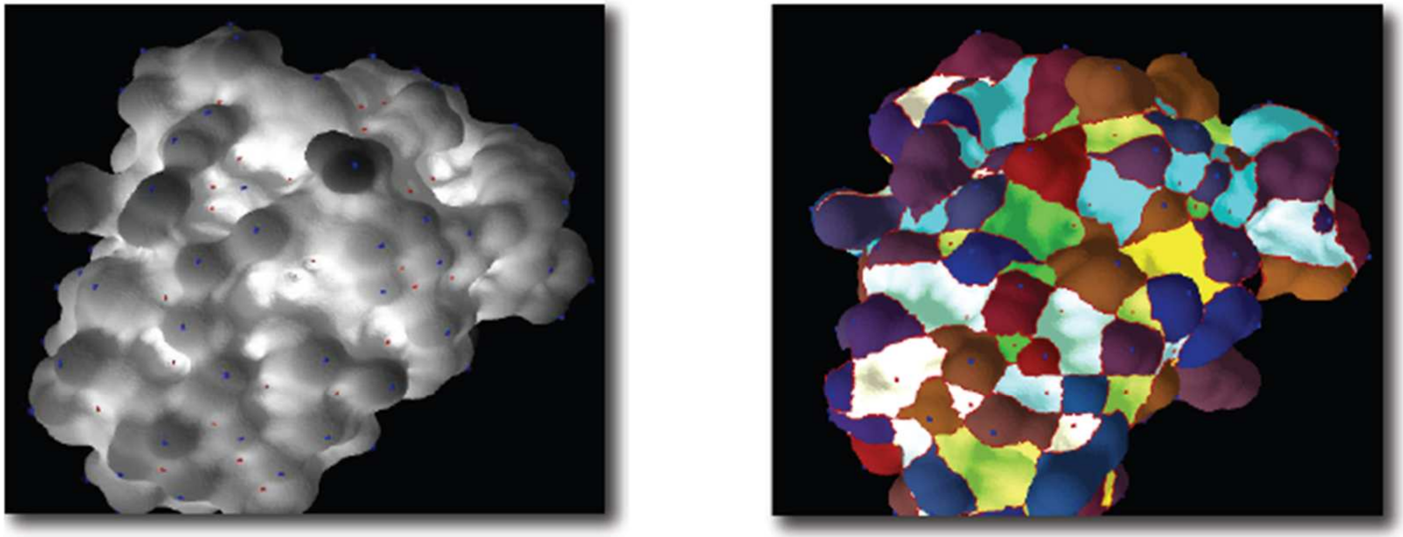


Application: Surface Segmentation

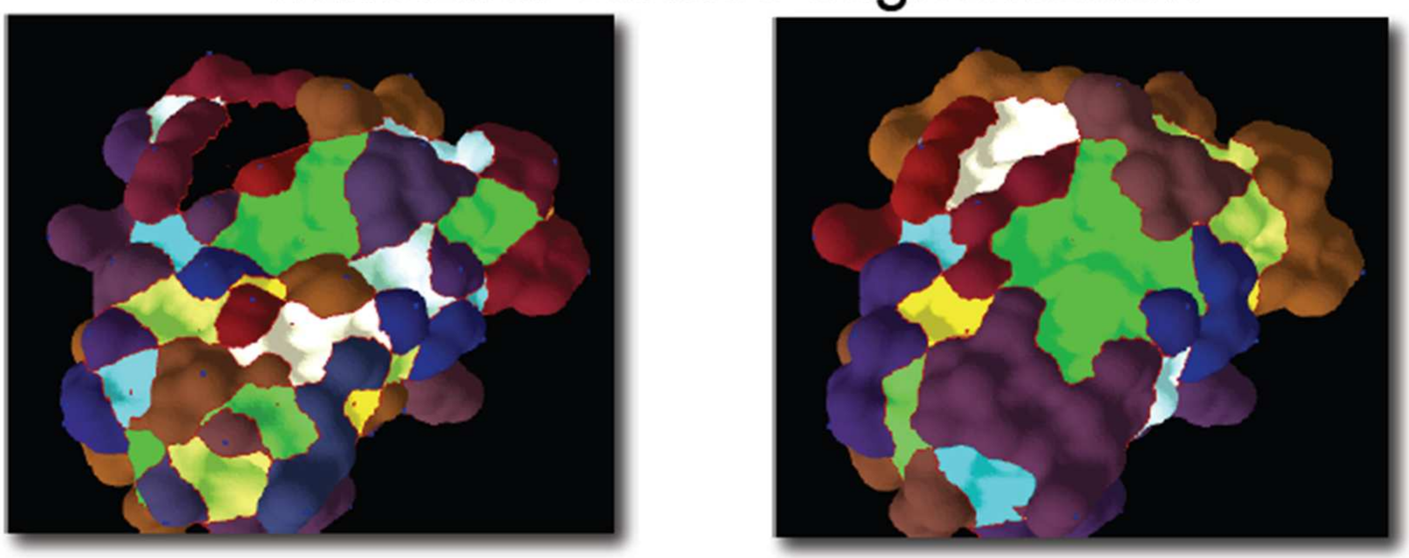
- Why segmentation?
  - Reduce the information overloaded
  - Identify unique features and properties

- There have been many proposed surface segmentation strategies to encode the structure of a function on a surface.
  - Surface networks ideally segment terrain-type data into the cells of the two-dimensional Morse-Smale complex, i.e., into regions of uniform gradient flow behavior. Such a segmentation of a surface would identify the features of a terrain such as peaks, saddles, dips, and the lines connecting them.
  - Image processing: watershed / distance field transform

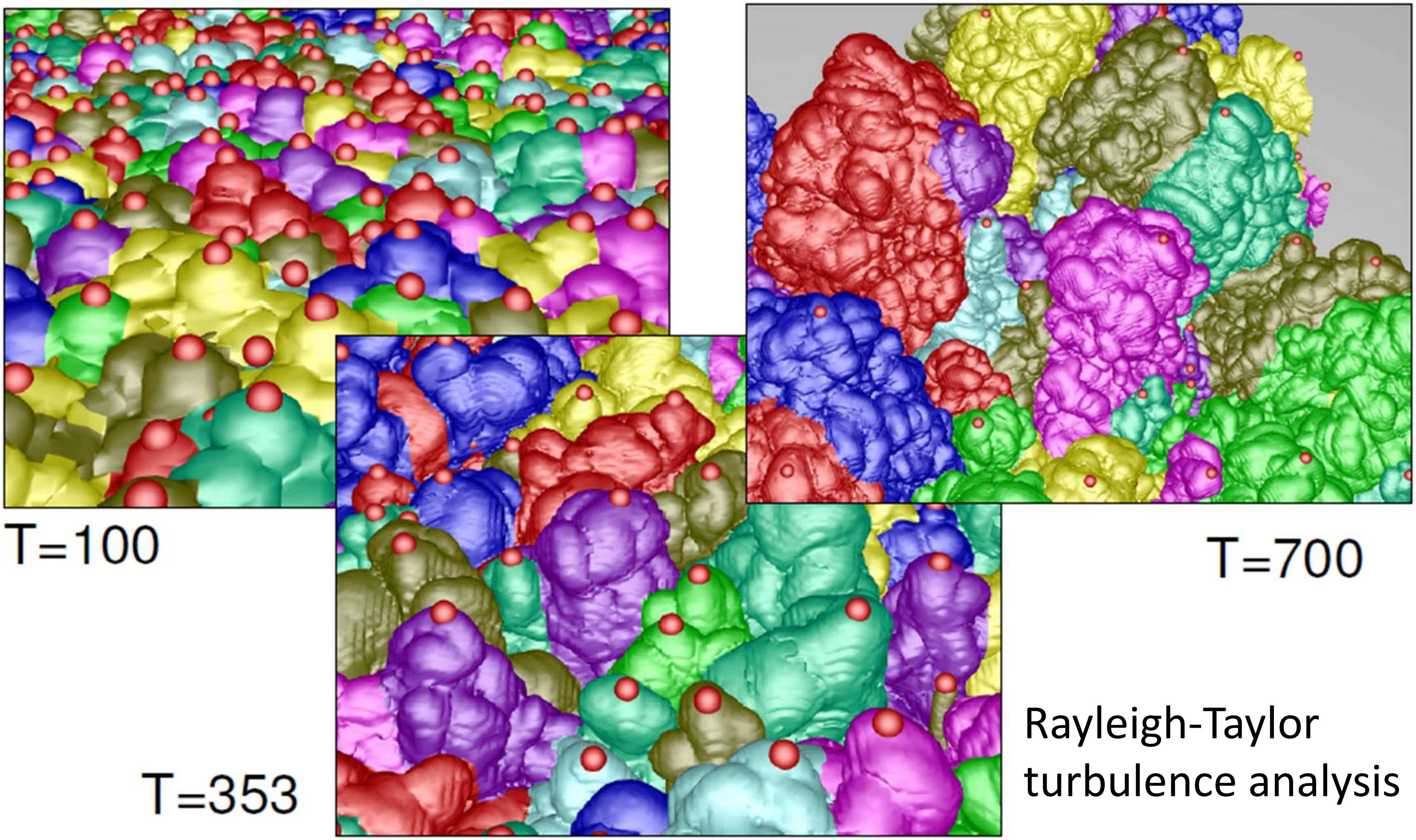
Applications



Molecular surface segmentation



Applications



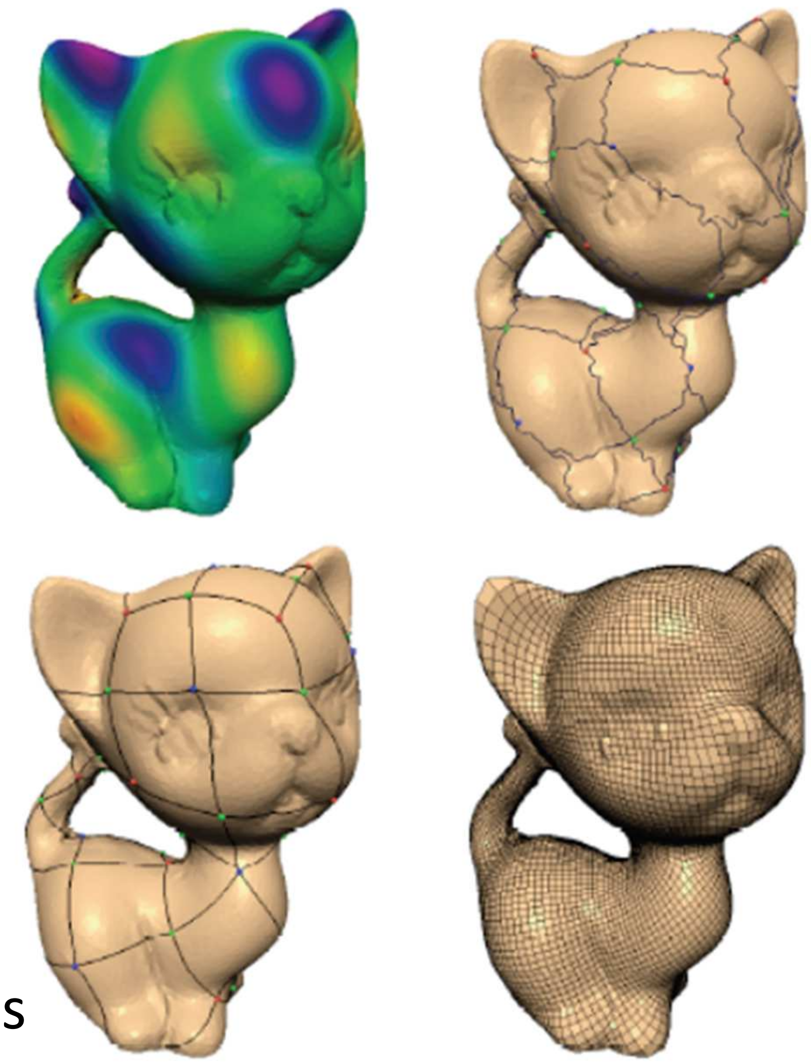
T=100

T=353

T=700

Rayleigh-Taylor turbulence analysis

Applications



Quadrangulation of surfaces

Additional Reading of M-S Complexes

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- H. Edelsbrunner, J. Harer, V. Natarajan, and V. Pascucci. “Morse-Smale complexes for piecewise linear 3-manifolds”. *Proc. ACM Symposium on Computational Geometry*, 2003, 361-370.
- P. T. Bremer, H. Edelsbrunner, B. Hamann and V. Pascucci. A Multi-resolution Data Structure for Two-dimensional Morse Functions. *IEEE Visualization*, 2003.
- H. Edelsbrunner, J. Harer and A. Zomorodian. Hierarchical Morse-Smale complexes for piecewise linear 2-manifolds. *Discrete and Computational Geometry,* 2003.
- A. Gyulassy, V. Natarajan, V.Pascucci, P.-Ti. Bremer and B. Hamann. “A topological approach to simplification of three-dimensional scalar fields”. Invited paper in *IEEE Transactions on Visualization and Computer Graphics*, 12(4), 2006, 474-484.
- A. Gyulassy, V. Natarajan, V. Pascucci, and B. Hamann. “Efficient computation of Morse-Smale complexes for three-dimensional scalar functions”. *IEEE Transactions on Visualization and Computer Graphics* (IEEE Visualization 2007), 13(6), 2007, 1440--1447.
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- S. Dong, P.-T. Bremer, M. Garland, V. Pascucci, and J. Hart, “Spectral surface quadrangulation”, ACM Transactions on Graphics, Volume 25 , Issue 3, pp.1057-1066qs (July 2006). Proceedings of *SIGGRAPH 2006.*

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