# An Evaluation of The Quality of Hexahedral Meshes Via Modal Analysis

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## Abstract

Hex-meshes have wide and important applications in scientific computing. Various methods have been proposed to tessellate a shape into a hex-mesh according to certain geometrical *quality criteria* (e.g. scaled Jacobian), which is usually not directly related to the downstream computing problem. This paper discusses such criteria on the elastic finite element analysis problem via analyzing the result of Modal Analysis performed on different hex-meshes. With the isometric uniform material configuration, the larger the scaled Jacobian (minimal/average) and the number of elements, the smaller are the eigenvalues, which usually indicates better accuracy and stability (condition number). Besides the element quality, singularity placement also affects the accuracy of the solution and is difficult to alter by simple subdivision. Results of this work may serve as a guide in evaluating hex-remeshing approaches and choosing appropriate hex-meshes for finite element analysis.

**Keywords:** Finite element analysis, hexahedral meshes, evaluation, singularity structure

## 1 Introduction

For many physically-based simulations involved in a wide range of scientific and engineering applications, volumetric representations, e.g., 3D volume tessellations, are required. Compare to the easily accessible tetrahedral meshes [1], hexahedral meshes are usually more attractive in their numerical property. But, generating a suitable hexahedral mesh conforming to the given boundary is typically more challenging than generating a tetrahedral one. In the past decades, various techniques for generating hexahedral meshes from the input boundary surfaces have been proposed [2, 3, 4]. Up to date, the meshing community still largely relies on Jacobians of the elements to evaluate the quality of the obtained meshes. That is, the meshes with larger average and minimum Jacobians are considered better. Indeed, the local Jacobian has been shown related to the local accuracy of the simulation [5]. There exist a number of studies on the benefits of conducting scientific simulations on hexahedral meshes versus tetrahedron meshes [6, 7, 8]. However, there still lacks of a comprehensive evaluation and comparison of the hex-meshes generated by recent methods from the application perspective. Further, there is little study on how different characteristics, such as the number of elements and different singularity structures of hex-meshes that are defined by non-valence-6 extraordinary nodes of the mesh (Figure 5), affect the simulations.

In this paper, we conduct a first systematic study on how various characteristics of the hex-meshes generated by a number of representative techniques affect the accuracy and stability of elastic object simulation. Particularly, the mesh characteristics that are considered in our study include the Jacobians of elements (i.e., average/minimum Jacobians), numbers of elements, and the structure of the hex-meshes (i.e., the numbers of singularities and the number of the hexahedral *components* in the structure). The hex-meshing methods that we are considering include Volumetric PolyCubes [9], SRF [3], and L1-based PolyCubes [4]. By selecting the models that are used in all these methods, we specifically focus on the rocker arm model-representing a CAD object and the bunny model-an example of organic objects. To evaluate the accuracy and stability of simulation run on different hex-meshes, *Modal Analysis* is employed. Especially, the eigenvalues and condition numbers of the stiffness matrices generated from the hex-meshes are measured and compared. It is worth emphasizing that this work aims to study what influence of the characteristics of different hex-meshes can have to the FEM simulations rather than identifying the best hex-meshing technique. However, results reported in this work can in turn be used to help practitioners select the proper mesh generation method given the metric that is sought by a specific computation.

## 2 Related Work

Over the past decades, several studies have been performed to compare the convergence behavior between the FEM based simulations run on tetrahedral, hexahedral, and hybrid meshes. Cifuentes et al. [10] concluded that the results obtained with quadratic tetrahedral elements, compared to bilinear hexahedral elements, were equivalent in terms of both accuracy of the simulated result and the computation time. The accuracy of elastic and elasto-plastic FEA based simulations of hexahedral and tetrahedral meshes was compared by Benzley et. al [6]. It showed a preference for linear hexahedral mesh compared to linear tetrahedral mesh. By evaluating the FEA simulations on Femur meshes, Ramos et al. [7] observed that the accuracy of the simulation conducted on a linear tetrahedral mesh of the simplified bone is closer to the theoretical ones, while the simulations on the quadratic hexahedral meshes are more stable to hex-meshes with higher resolution.

Up to date, there exists little study on comparing the accuracy properties of simulations computed on the hex-meshes generated with recent techniques. Earlier, Muller et al.[11] investigated the effect of quality and the number of elements of hex-meshes on the simulation results, respectively. They concluded that the number of elements of hex-meshes is important for the simulation quality. However, their experimental results suggest that with the minimum element quality above 0.1, average mesh quality does not has notable impact on the solution accuracy. Since they only used a very simple cylinder-like model throughout the whole experiments, it is difficult to judge the completeness of their study. In this work, we propose to employ a few relatively complex models to verify their statements. Our study is also the first one that attempts to analyze the impact of the singularity structures of hex-meshes on the simulation solutions.

## 3 Comparison Experiment Setup

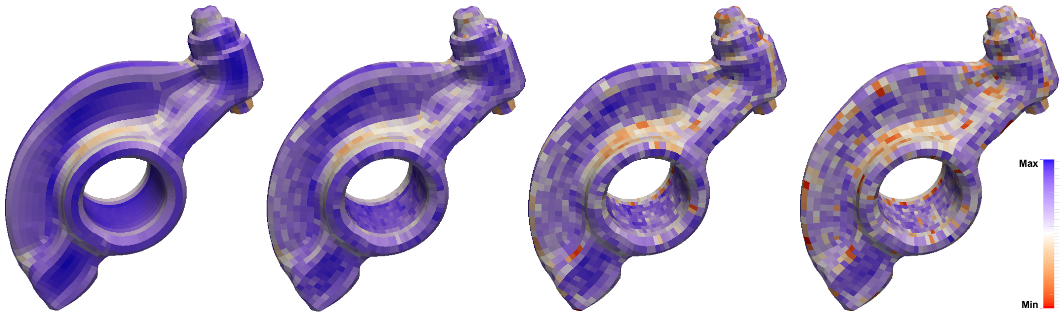
The focus of this paper is to investigate the accuracy and convergence rate of simulations performed on the selected hex-meshes that have (1) varying local qualities (measured by Jacobians), (2) different numbers of elements, and (3) different singularity structures (measured by different numbers). In order to conduct such a comparison, several issues have to be addressed, including “which finite element method to use?”, “how to measure the dissimilarity between two simulations?”, and “which 3D models to use?”. These topics are discussed in the following.

### *3.1 Modal Analysis of Elastic Problem*

To understand the impact of different properties of the hex-meshes on the FEA based simulations, we make use of a well studied technique known as Modal Analysis (MA). This section presents a brief overview of modal analysis. For more details, please refer to [12].

Giving an 3-dimensional linear elastic object discretized by a mesh with $n$ nodes, equation of motion is often formulated as $M\ddot{u} + Ku = f$, where $K, M \in \mathbb{R}^{3n\times 3n}, u, f \in \mathbb{R}^{3n}$ are the stiffness matrix mass matrix, node displacement and external force respectively. Modal Analysis computes the general eigen problem $W^TKW = \Lambda, W^TMW = I$, and then which diagonalizes the above equation into $\ddot{z}+\Lambda z = g$ via substitution $u = Wz, g = W^Tf$. Such a transform does not only boost the efficiency of computation, but reveal an important property of the dynamics: the motion can be decomposed into a set of independent vibration modes (columns of $W = (W_1, W_2, \cdots, W_n)$) at their stiffness (square of natural frequency) $\lambda_i = \Lambda_{ii}, \lambda_i \leq \lambda_{i+1}$.

Eigenvalue is a widely used indication for the quality of discretization in terms of accuracy and stability in computation. As explained




$(a)0.916/0.209$ $(b)0.897/0.118$ $(c)0.860/0.100$ $(d)0.828/0.100$

Figure 1: Original rocker arm hex-mesh from [3] and the generated hex-meshes with varying quality.

in [6], the discretization introduces additional numerical stiffness because of the extra force to restrict the deformation in a small subspace. The worse discretization is, the eigenvalue is more overestimated, and the object is "stiffer". Thus, smaller eigenvalue usually indicates better accuracy in discretization. In many cases, the condition number of the linear system involved in the simulation is nearly determined by $\max\{\lambda_i\}$. For example, an equation similar to $(M + h^2K)\Delta v = hf$ should be solved in each step of backward Euler method [13] for the change of velocity $\Delta v$ in time step $h$. The minimum and maximum eigenvalues of the system $(M + h^2K)$ (respect to $W$) are 1 and $1 + h^2 \max\{\lambda_i\}$ respectively, because the leading 6 eigenvalues associated with rigid motion are all zero. Thus, the smaller the eigenvalue is and the smaller the condition number is, the better stability the simulation will have.

The discretization is characterized by the mesh tessellating the object and the type of basis applied to the elements in the mesh. In this paper, we adopt 8-node trilinear basis function for all the hexahedral elements, and thus leave the properties (e.g. Jacobian, density) of mesh as the only factor related to the eigenvalues.

### 3.2 3D Model Data

Table 1 provides an overview the hex-meshes used for different comparison studies. Among them, different versions of $Bunny^1$ from method [9] and $Bunny^2$ from [3] are used to examine the Modal Analysis properties of hex-meshes with various element numbers, while the first three versions of $Rockerarm^1$ [4] and $Rockerarm^2$ [4] are employed for the analysis on meshes with different scaled Jacobians (i.e., Ave./ Min. Jacobians). Three hex-meshes, denoted by $Rockerarm^1_3$ [4], $Rockerarm^2_3$ [4],

Table 1: Meshes used in the three Comparisons

| Hex-Mesh | V/H | Ave./Min J. |
|---|---|---|
| $Bunny^1_0$ | 59887/54568 | 0.943/0.136 |
| $Bunny^1_1$ | 49524/44909 | 0.941/0.128 |
| $Bunny^1_2$ | 41593/37511 | 0.937/0.127 |
| $Bunny^1_3$ | 35428/31759 | 0.940/0.125 |
| $Bunny^2_0$ | 51175/47797 | 0.947/0.316 |
| $Bunny^2_1$ | 32410/30119 | 0.948/0.322 |
| $Bunny^2_2$ | 18654/17226 | 0.944/0.279 |
| $Rockerarm^1_0$ | 28397/24346 | 0.930/0.123 |
| $Rockerarm^1_1$ | 28397/24346 | 0.894/0.102 |
| $Rockerarm^1_2$ | 28397/24346 | 0.862/0.100 |
| $Rockerarm^2_0$ | 28814/24780 | 0.916/0.110 |
| $Rockerarm^2_1$ | 28814/24780 | 0.881/0.102 |
| $Rockerarm^2_2$ | 28814/24780 | 0.849/0.100 |
| $Rockerarm^1_3$ | 12919/10776 | 0.938/0.413 |
| $Rockerarm^2_3$ | 13014/10942 | 0.926/0.457 |
| $Rockerarm^3$ | 12751/10600 | 0.916/0.209 |

and $Rockerarm^3$ [3], with distinct singularity structures are also selected to investigate the influence of the structure on the FEA. Note, that the sup-index of a mesh, e.g., $Rockerarm^3$, represent one of the three methods [9, 4, 3] that is used to generate the mesh, while sub-indices, e.g., $Bunny^1_0$ and $Bunny^1_1$, indicate different versions of the meshes. These hex-meshes are either directly from the three work or generated from the original hex-meshes using the follow methods.

To produce hex-meshes that have the same number of elements and singularity structures, but varying local quality, we follow the procedure in [11]. The difference of our approach is that for the vertex $v_i$ that is to be jittered, a bounding surface is also constructed to constrain the active space of $v_i$. The bounding surface is comprised of the quads of the one-ring hexahedra of $v_i$ that are not adjacent to $v_i$. By adding this constraint, the possibility of generating inverted elements is excluded. Figure 1 shows the generated rocker arm meshes with different Jacobians.

Simplification technique of [14] is adapted to generate meshes with different numbers of elements . All simplified meshes are optimized by Mesquite to achieve similar Jacobian quality. To consider the effect of different singularity structures of the meshes to the Model Analysis, the rocker arm meshes from two methods [4, 3] are employed, which already have different structures (Figure 5). We use the same approach above to ensure they have similar numbers of elements and similar Jacobians.

# 4 Results

To evaluate the quality of hex-meshes generated by different approaches, three types of comparisons have been performed, including the comparisons of hex-meshes with 1) different Ave./Min. Jacobians, 2) various numbers of elements, and 3) distinct singularity-structures. The quality of the mesh is measured by the $i$th eigenvalue $\lambda_i^m$ for mesh $m$.

In Figure 2, we investigate the meshes with the same number of elements. To better visualize the differences among the eigenvalues derived from different meshes, we normalize $\lambda_i^m$ into $\tilde{\lambda}_i^m = (M\lambda_i^m)/\sum_{j=1}^{M}\lambda_i^j$, where $M$ is the total number of meshes in the comparison. The smaller normalized value $\tilde{\lambda}_i^m$ is, the smaller the eigenvalue $\lambda_i^m$ will be. Figure 2 demonstrates that as the average scaled Jacobians of $Rockerarm^1$ (Figure 2(a)) and $Rockerarm^2$ (Figure 2(b)) increase, the eigenvalues of the hex-meshes decrease. This observation confirms that better Jacobians result in better accuracy and condition number.

In Figure 3(a) and (b) we compare the meshes with similar scaled Jacobians. As shown in Figure 3(a), when the number of elements has no big difference (in the range of 35k to 60k), the improvement of accuracy by using more elements is not significant. But when the density increases a lot (from 19k to 51k), the improvement becomes obvious.

As shown in Figures 4 and 5, the hex-meshes have very distinct singularity structures. The distribution of the deformation displacements and the eigenvalues for different modes are also different. From Figure 4, although $Rockerarm_3^1$ and $Rockerarm_3^2$ have better Ave./Min. Jacobians and larger number of elements, $Rockerarm^3$ gives the best eigenvalues for all modes. This may indicate that not only the scaled jacobian and the element number of a hex-mesh can affect its finite element analysis, the singularity-structure of a hex-mesh can also make an influence. Figures 4 shows that the numbers of singularity nodes and hex-patches of $Rockerarm^3$ are ranked in between of those of $Rockerarm_3^1$ and $Rockerarm_3^2$. By comparing with their corresponding eigenvalues in Figure 4, it implies that besides the numbers of singularity nodes and hex-patches, there could be other properties of the singularity-structure that affect its FEA behavior.

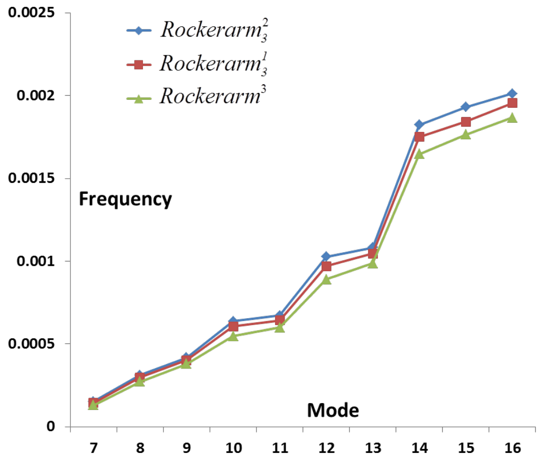
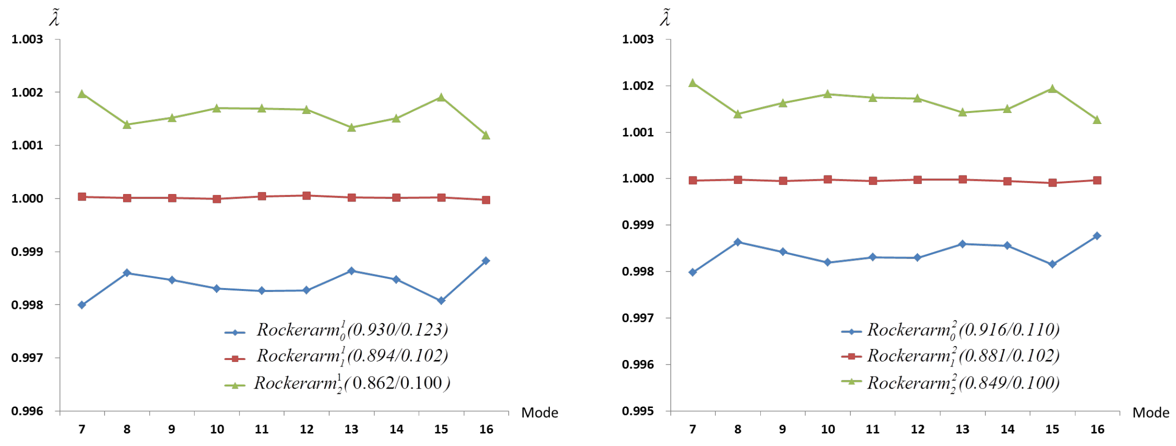



Figure 4: Comparisons among the eigenvalues corresponding to various modes for three rocker arm meshes.

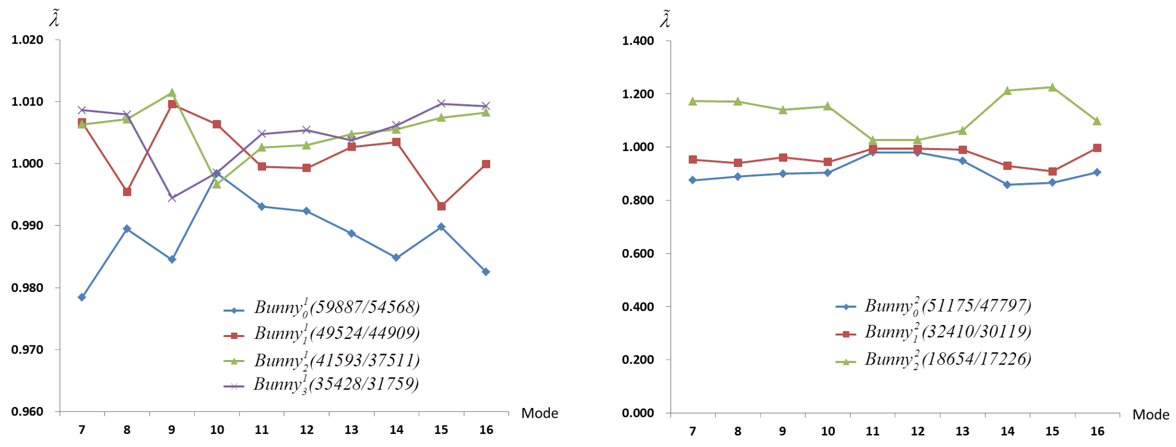
# 5 Conclusion and Future Work

In this paper, we investigate how different characteristics of hex-meshes affect simulations conducted on these meshes. In particular, Model Analysis is performed on the stiffness matrices computed on these meshes. Hex-meshes for a rocker arm and a bunny with different characteristics (e.g., different numbers of elements, different Jacobians, and different singularity structures) are utilized for this study. Our initial results indicate that when hex-meshes have the same singularity-structure, the better behavior of Modal Analysis is typically observed on hex-meshes with larger scaled Jacobians and more hex-elements. When the hex-meshes have distinct structures but similar scaled Jacobians and




(a) (b)

Figure 2: Comparisons among hex-meshes with various Jacobians denoted in the legends.




(a) (b)

Figure 3: Comparisons among hex-meshes with various numbers of elements denoted in the legends.

element numbers, the performances of Modal analysis can still vary. In particular, good singularity-structures (e.g., with smaller number of singularity nodes and hexahedral component) could compensate the shortness of smaller Jacobians or fewer elements.

**Limitations and Future Work** There are some limitations in the current study. First, only two objects are used in the experiments, which may not be convincing. Second, some properties like the distribution of elements quality, and the distribution of different scales of elements are not covered in the current study. Third, how the difference of singularity structures should be defined? This is still an open problem. Also, the formal relation of the singularity structure with the accuracy and performance of the simulations needs to be studied rigorously. Fourth, more hex-meshing approaches should be included. Finally, other simulations than elastic deformation, such as Isogeometric Analysis (IGA) [15], should be examined in order to fully evaluate the benefits of structured hex-meshes.

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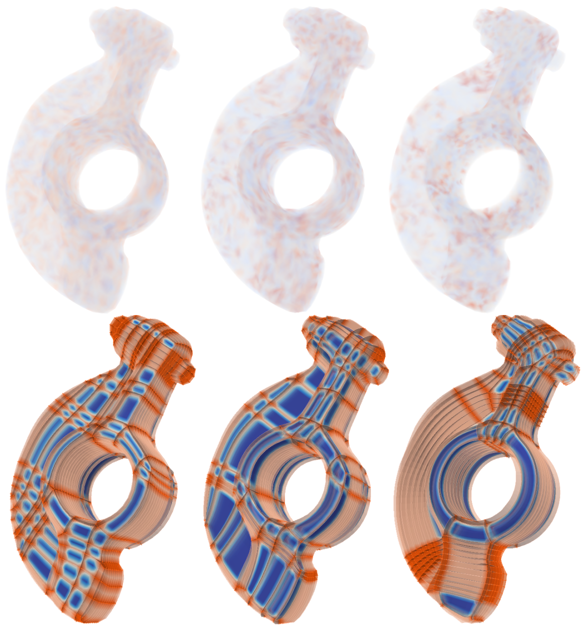

$(a)88/1856$ $(b)64/948$ $(c)80/1149$

Figure 5: Rocker arm hex-meshes produced by (a-b) [4] and (c) [3]. First row shows the displacement distribution, second row shows the corresponding singularity-structures for them. The numbers of singularity nodes and the patch numbers of the singularity-structures for these meshes are 88/1856, 64/948, 80/1149, respectively.


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