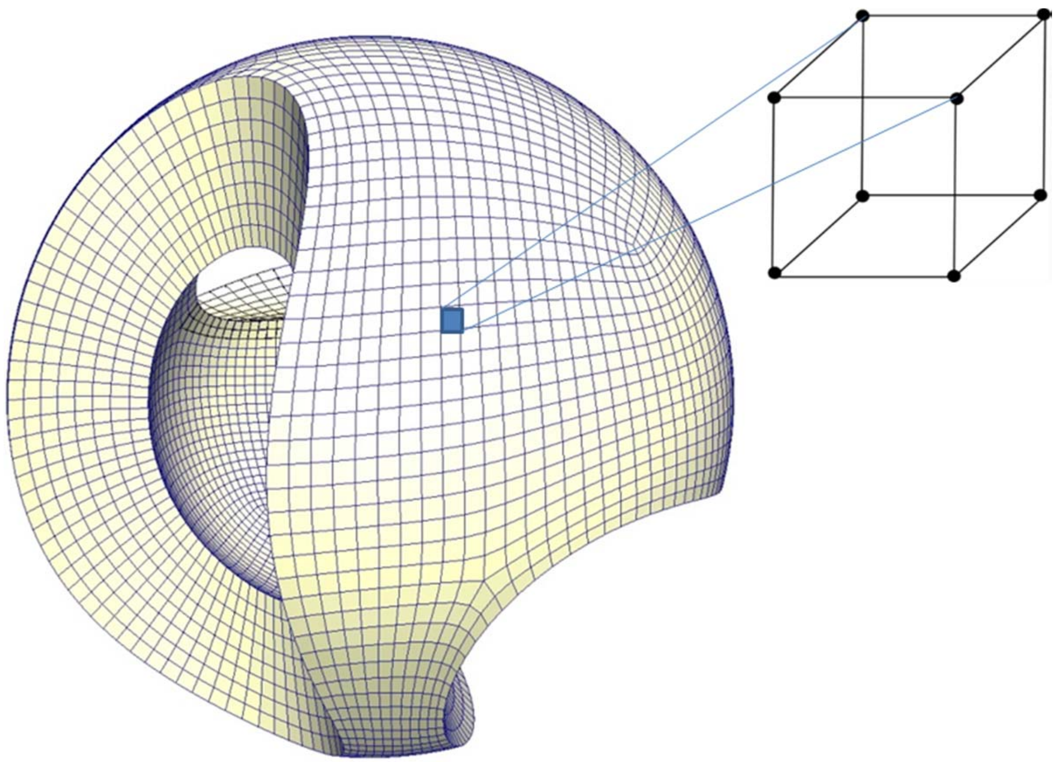
Evaluating Hex-mesh Quality Metrics via Correlation Analysis

Xifeng Gao[1] , Jin Huang[2], Kaoji Xu[3], Zherong Pan[4], Zhigang Deng[3], Guoning Chen [3]

[1] New York University
[2] Zhejiang University
[3] University of Houston
[4] University of North Carolina at Chapel Hill

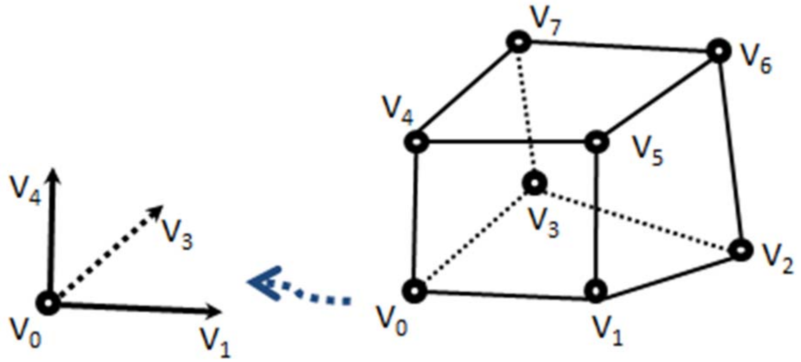
What is A Hexahedral (Hex-) Mesh?



Quality metrics for Hexahedra

$$\det(A_0) = \overrightarrow{v_1 v_0} \cdot (\overrightarrow{v_3 v_0} \times \overrightarrow{v_4 v_0})$$

$$J_{jacobian} = \min\{\det(A_i), i = 0, 1, 2 \cdots, 7\}$$

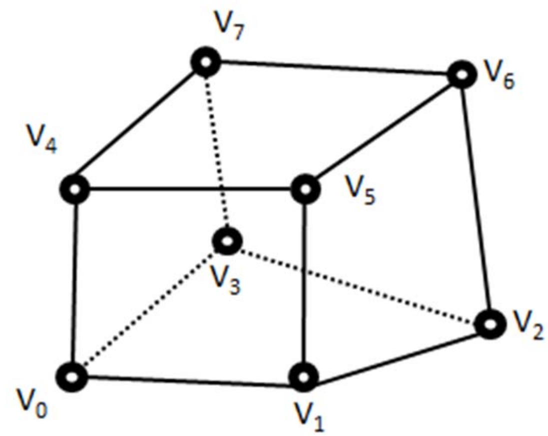


$$\det(\hat{A}_0) = \frac{\overrightarrow{v_1 v_0}}{\|\overrightarrow{v_1 v_0}\|} \cdot \left(\frac{\overrightarrow{v_3 v_0}}{\|\overrightarrow{v_3 v_0}\|} \times \frac{\overrightarrow{v_4 v_0}}{\|\overrightarrow{v_4 v_0}\|}\right)$$

$$S.J_{scaled\_jacobian} = \min\{\det(\hat{A}_i), i = 0, 1, 2 \cdots, 7\}$$

Quality metrics for Hexahedra and Hex-Meshes

$$Diagonal = \frac{D_{\min}}{D_{\max}} \qquad Ratio_{edge} = \frac{L_{\max}}{L_{\min}}$$



- Extreme (maximum/minimum) metric is to locate the element in the hex-mesh with the worst quality;

- Average metric is to compute an averaged quality of all the elements of the hex-mesh.

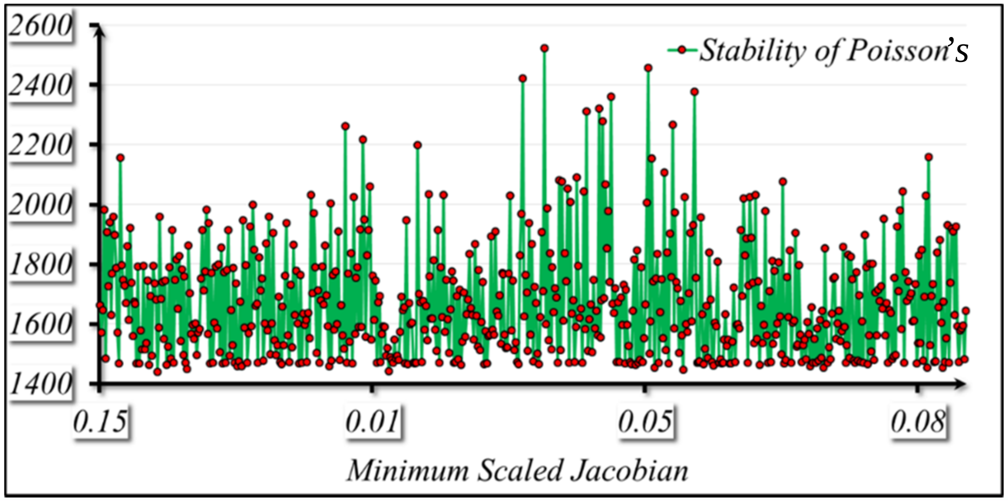
Quality metrics for Hex-Meshing

Metric	Abbr.	Range	Range*	Trend
diagonal	D.	[0, 1]	[0, 1]	↑
dimension	DM.	$[0, +\infty]$	$[0, +\infty]$	↑
distortion	DIS.	$[-\infty, +\infty]$	[0, 1]	↑
edge ratio	ER.	$[1, +\infty]$	$[1, +\infty]$	↓
Jacobian	J.	$[-\infty, +\infty]$	$[0, +\infty]$	↑
maximum edge ratio	MER.	$[1, +\infty]$	$[1, +\infty]$	↓
aspect Frobenius	AF.	$[1, +\infty]$	$[1, +\infty]$	↓
mean aspect Frobenius	MAF.	$[1, +\infty]$	$[1, +\infty]$	↓
Oddy	O.	$[0, +\infty]$	$[0, +\infty]$	↓
relative size squared	RSS.	[0, 1]	[0, 1]	↑
scaled Jacobian	S. J.	[-1, 1]	[0, 1]	↑
shape	S.	[0, 1]	[0, 1]	↑
shape size	SS.	[0, 1]	[0, 1]	↑
shear	SE.	[0, 1]	[0, 1]	↑
shear size	SES.	[0, 1]	[0, 1]	↑
skew	SK.	[0, 1]	[0, 1]	↓
stretch	ST.	[0, 1]	[0, 1]	↑
taper	T.	$[0, +\infty]$	$[0, +\infty]$	↓
volume	V.	$[-\infty, +\infty]$	$[0, +\infty]$	–

From a Sandia report on “the verdict geometric quality library”

Problem

- Given these many metrics, which one should we use to measure the quality of a given hex-mesh in practice?
- A hex-mesh with a positive minimum scaled Jacobian is a hard requirement to conduct PDE-based simulations.
- Once this minimum requirement is satisfied, however, is the scaled Jacobian still the most effective quality indicator for a hex-mesh?



Problem

- This asks for **a comprehensive study** of the relations among various quality metrics for hex-meshes.

Challenge

- However, there is no a clear mathematical formulae that can describe the relations among different features

$$q = \frac{L_{\max}}{L_{\min}} \quad \overset{f??}{\sim} \quad q = \min\left\{ \{\alpha_i\}_{i=0}^{7}, \frac{\alpha_8}{64} \right\}$$

Our Solution

We look at the co-variant behaviors among metrics!

We look at the quality of the entire mesh rather than a single element!

Before this Work

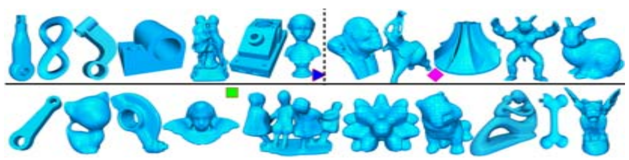
- Comparisons of tet-meshes and hex-meshes [CK92,BPM95,RS06,BTPB07, TEC10, TEC11, Cha13]

- Knupp [Knu00b] and Muller et al. [MHKSZ01] concluded that both of the number of hex-mesh elements and the average scaled Jacobian have positive impacts on the convergence and the accuracy of the simulations

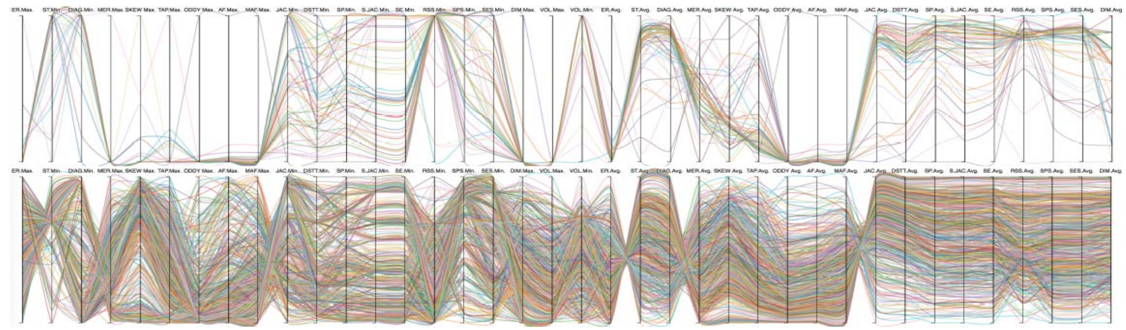
- Motooka et al. [MNI11] concluded that the diagonal length ratio affected the convergence of the Poisson’s equation solving.

Pipeline

Data Generation

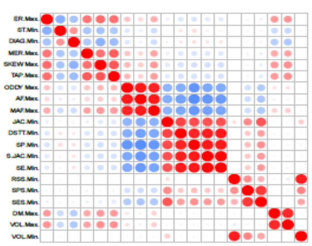


Perturbation

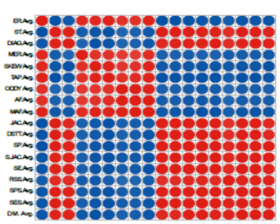


Correlation Analysis

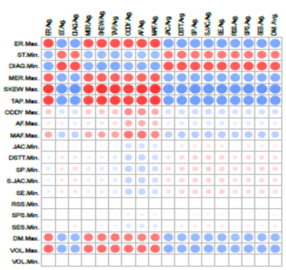
Discoveries



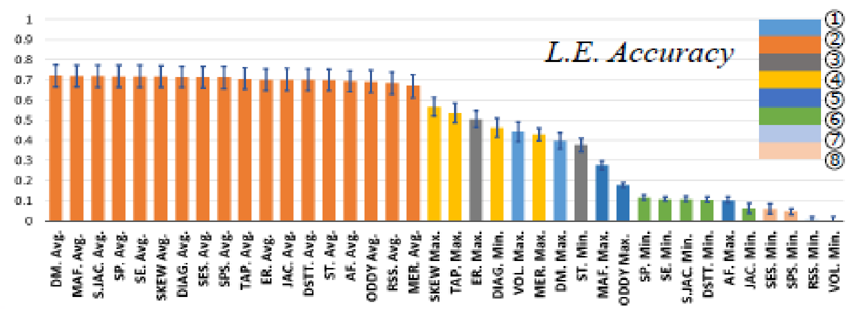
(a) Average correlation matrix of Min/Max metrics



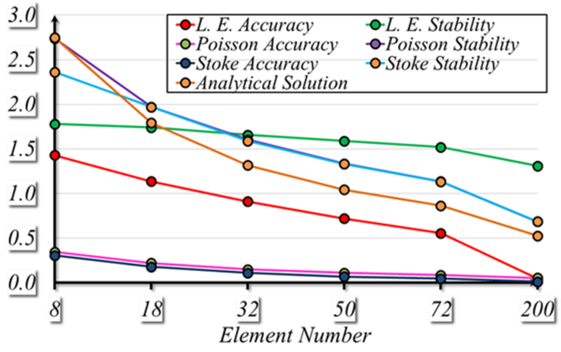
(b) Average correlation matrix of Avg. metrics



(c) Average correlation matrix of Min/Max metrics and Avg. metrics



Data Generation

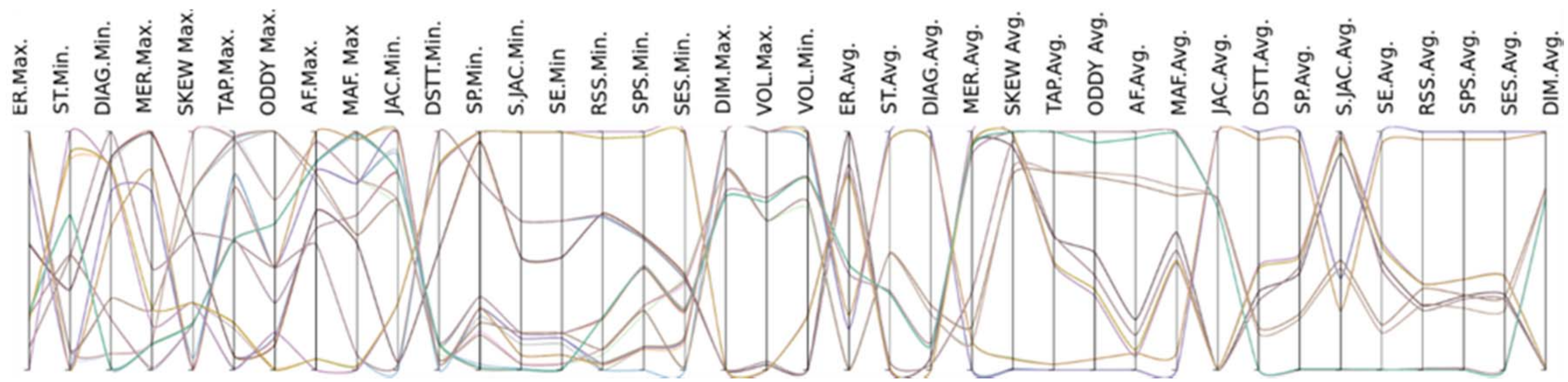


Requirements

- All hex-meshes in the same dataset should have the same number of elements and volume;
- *Only include valid meshes (i.e., the minimum scaled Jacobian is positive);*
- The dataset should cover a value range for each quality metric as wide as possible (i.e. having enough variation).

Data Generation

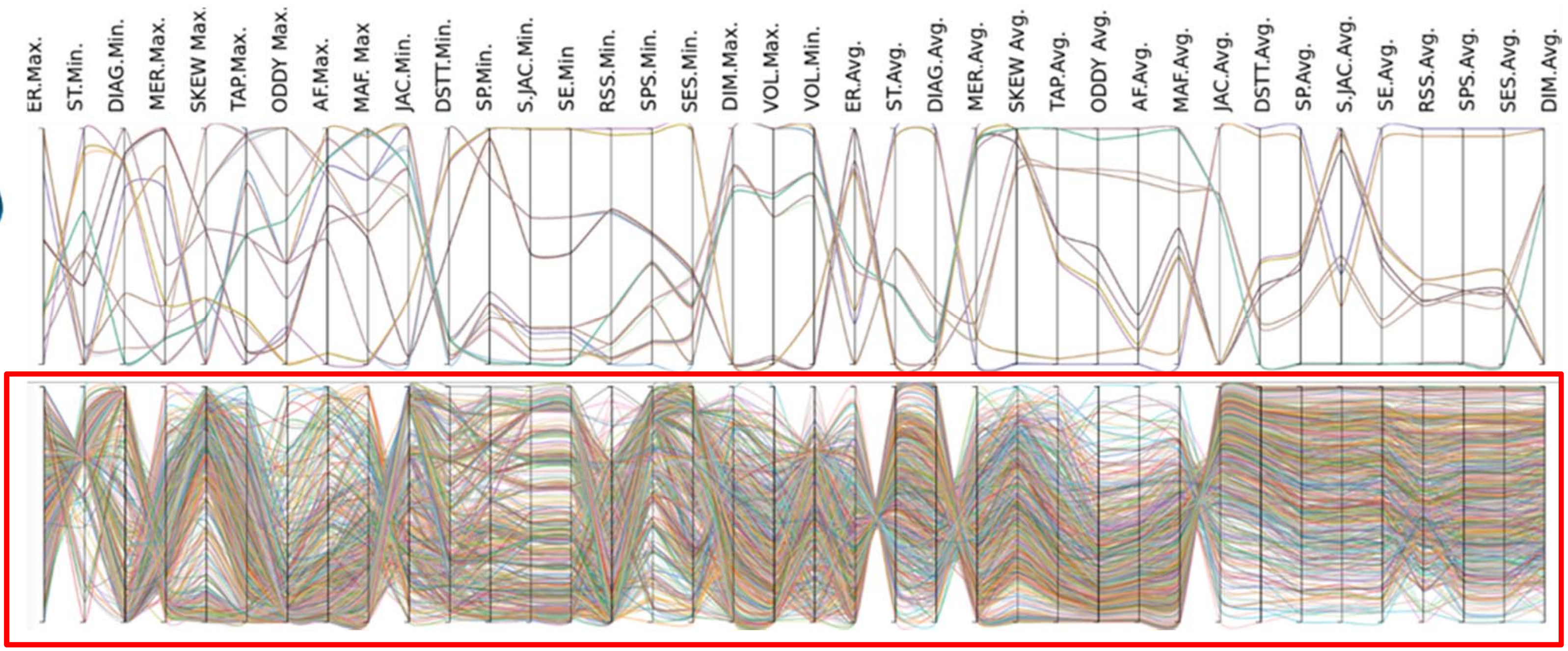
Problem with the dataset produced by the available hex-meshing tool -- MeshGem



More than 600 meshes were generated!

Data Generation

Our solution -- a two-level noise insertion



Data Generation

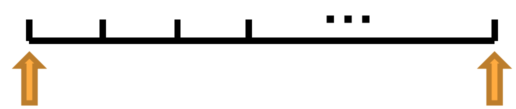
Our solution -- a two-level noise insertion (or perturbation)

The best available hex-mesh for the model

A hex-mesh input

Datasets by disturbing a vertex to change minimum metrics

Meshes generated by polycube-based and frame field based methods.



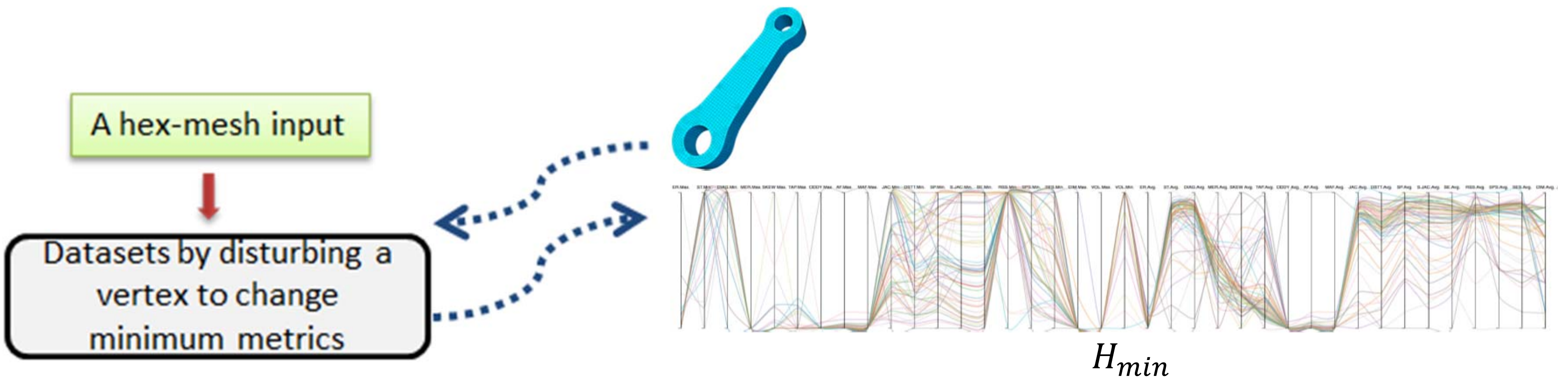
Min_bound

Min_value of the input mesh

Number of bins $\alpha = 8$

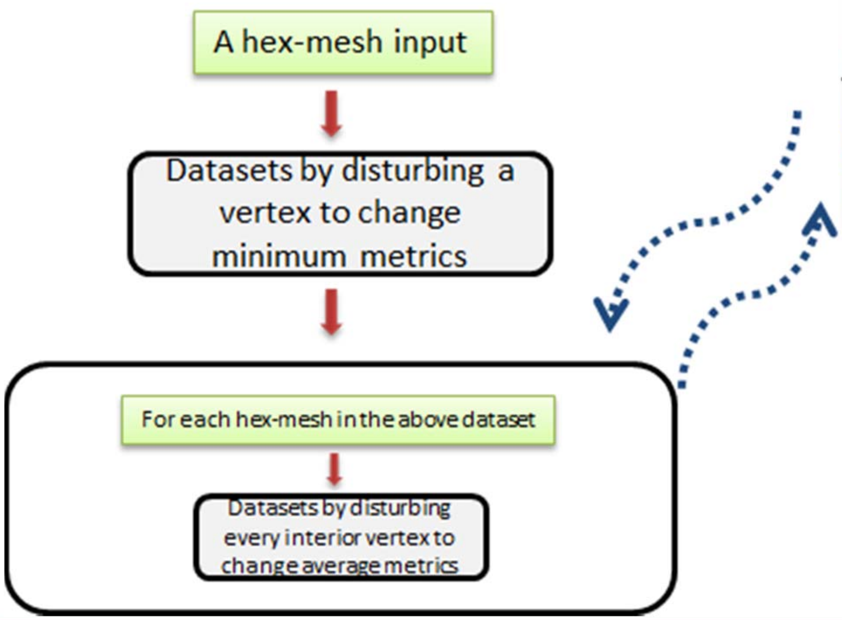
Data Generation

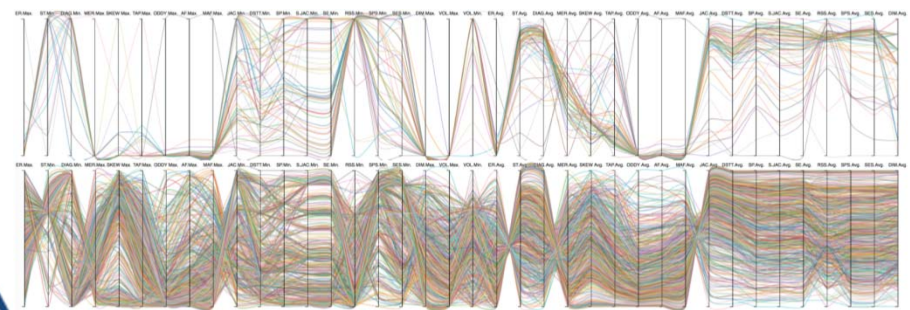
Our solution -- a two-level noise insertion



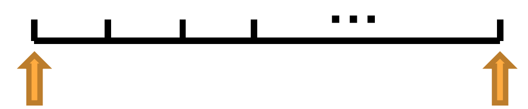
Data Generation

Our solution -- a two-level noise insertion





H_{avg}



Min_bound

Avg_value of the input mesh

$\beta = 20$

Data Generation

Table 2: *Statistics of six datasets. #H represents the number of elements in a mesh. $\mathcal{H}_{Min}$, $\mathcal{H}_{Avg}$ and $\mathcal{F}_{Avg}$ indicate the number of hex-meshes generated in different stages of the data generation, respectively. Timing shows the time for data generation, while Timing* is the total time for simulations.*

Datasets	Bone	Bust	Elephant	Hanger	Bunny	Rockerarm
#H	3396	5398	8730	4539	4552	5993
$\mathcal{H}_{Min}$	64	57	49	62	60	59
$\mathcal{H}_{Avg}$	3724	2752	3795	3634	3903	4554
$\mathcal{F}_{Avg}$	525	607	643	719	735	565
Timing	24h	40h	4h	51h	34h	61h
Timing*	8.3h	10.5h	41.3h	10.5h	8.5h	12.5h

Correlation Study

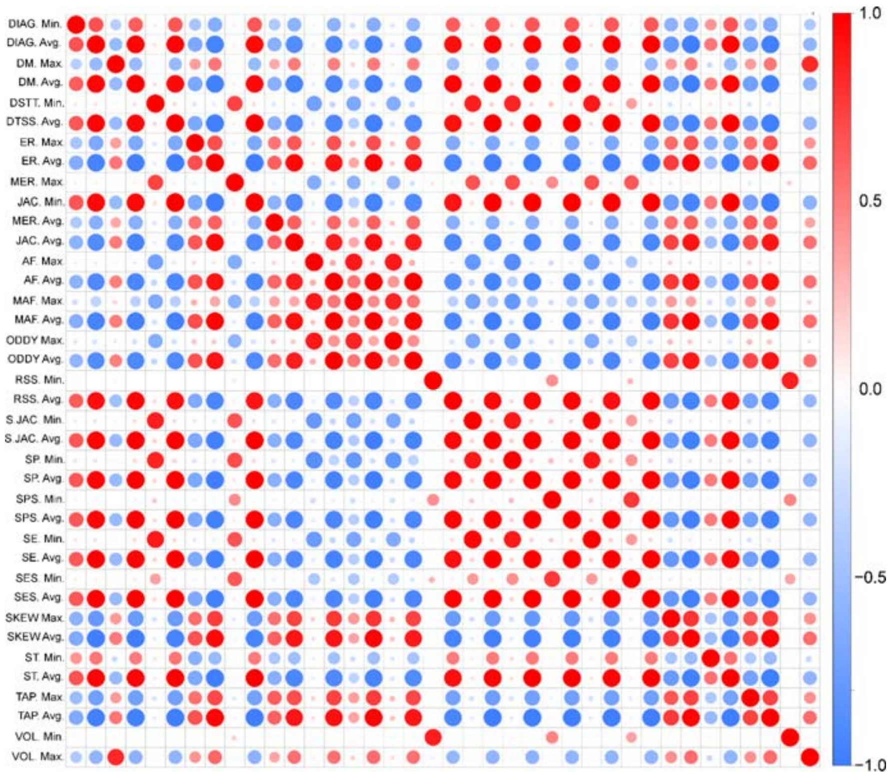
- Linear correlation coefficient
  [Sti89]

$$r_{x,y} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$$

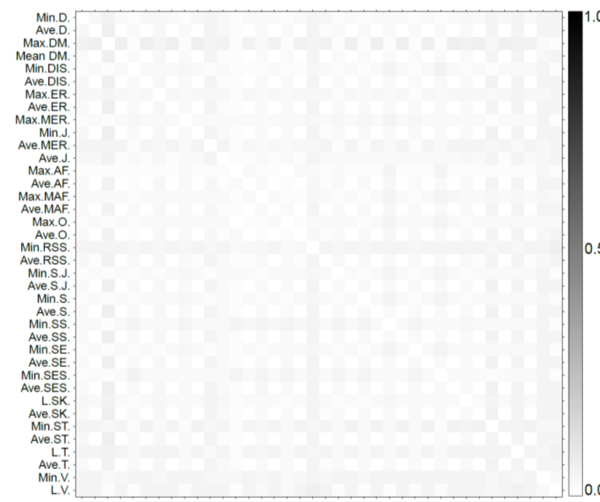
- Correlation matrix

$$C = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,N} \\ r_{2,1} & r_{2,1} & \cdots & r_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ r_{N,1} & r_{N,2} & \cdots & r_{N,N} \end{bmatrix}$$

Metric Correlation



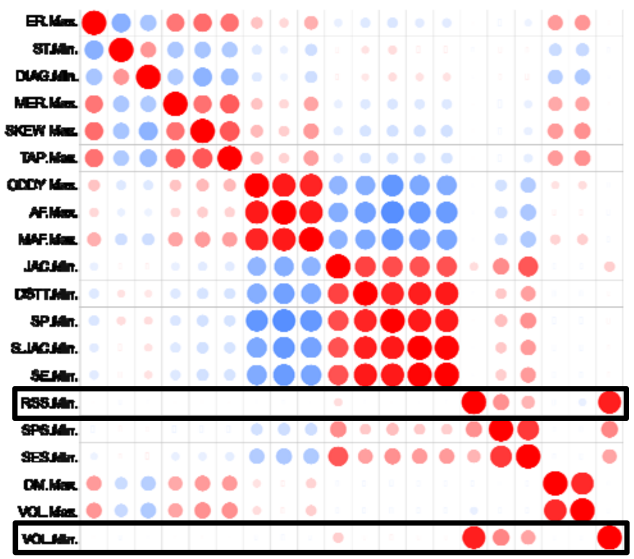
Average correlation matrix



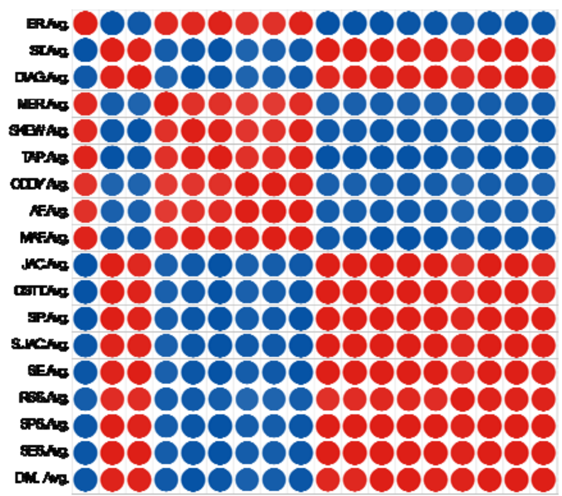
Deviation matrix

Application-Independent Study

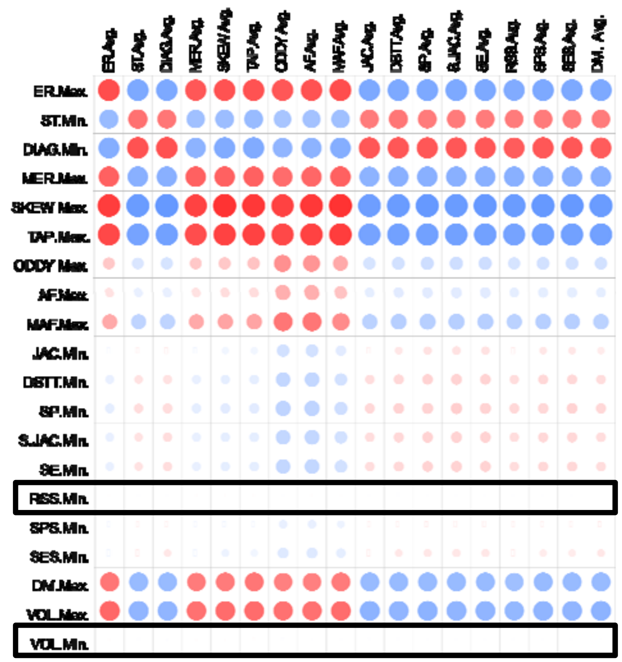
Metric Correlation



Min/Max Metrics



Average Metrics

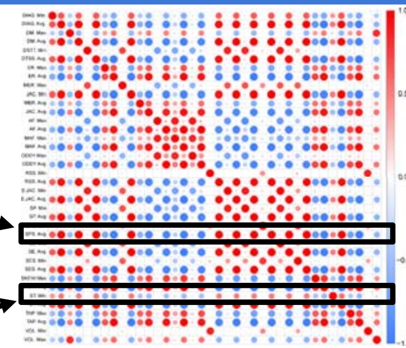


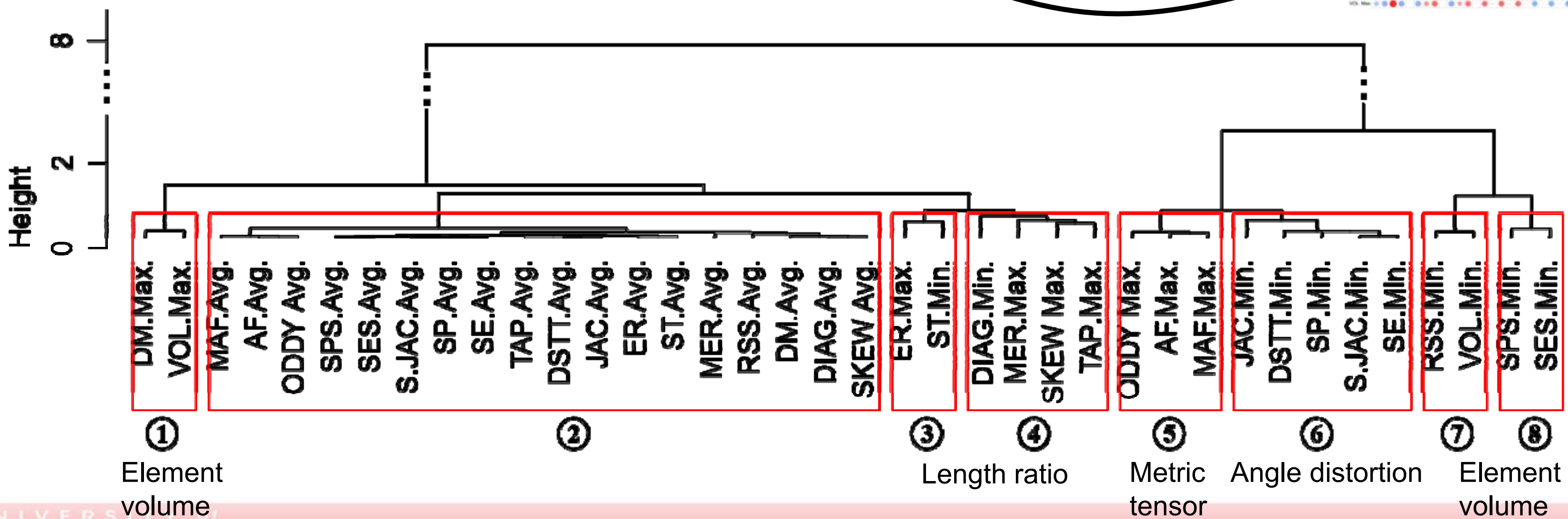
Min/Max and Average Metrics

Application-Independent Metric Correlation

Metric clustering based on their similarity [Ward's approach]

$$d(\vec{i},\vec{j}) = \frac{|\vec{i}|\,|\vec{j}|}{|\vec{i}|+|\vec{j}|}\left\|\vec{i}-\vec{j}\right\|^2$$

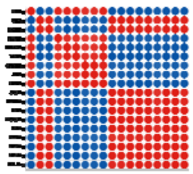




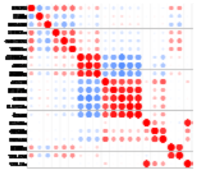
Application-Independent Metric Correlation

Observations

- All average metrics have strong correlations with each other -> any average metric is enough to measure the overall quality of a hex-mesh;



- Extreme metrics have different correlations according to different geometric characteristics.



Application-Dependent Study

Application-Dependent Metric Correlation

Simulations involved the solving of various elliptic PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

$Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is positive definite

- The linear elasticity
- Poisson's Equation
- Stokes Equation
- An analytic problem with known solution

Application-Dependent Metric Correlation

Measuring the quality of solving elliptic PDEs on discrete spatial representation is challenging **without** knowing the **ground truth**.

We propose to employ the maximum and minimum eigenvalues as the measurement for simulation quality. [BPM95] [She02]

- Minimum leading eigenvalue λ_{min} -- indication of discretization error
  Accuracy metric: minimum eigenvalue minus ground truth

- Maximum eigenvalue λ_{max} -- related to the conditioning of the system
  Stability metric: the condition number $\lambda_{max}/\lambda_{min}$

Application-Dependent Metric Correlation

Simulations involved the solving of various elliptic PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

- The linear elasticity
- Poisson's Equation
- Stokes Equation
- An analytic problem with known solution

$Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is positive definite

The quality is measured by the difference between the solved value and the ground truth.

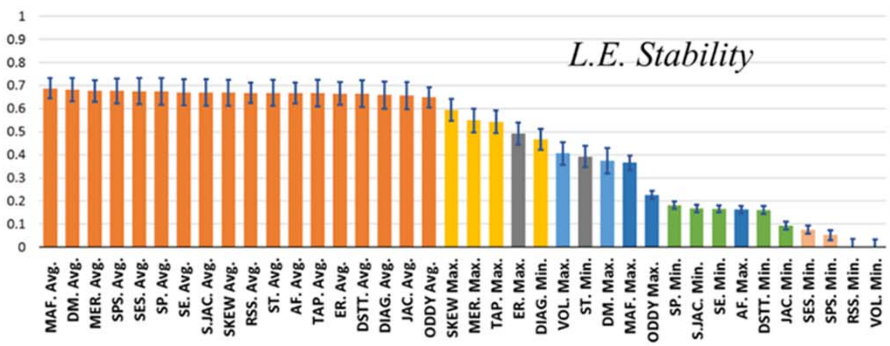
Application-Dependent Metric Correlation

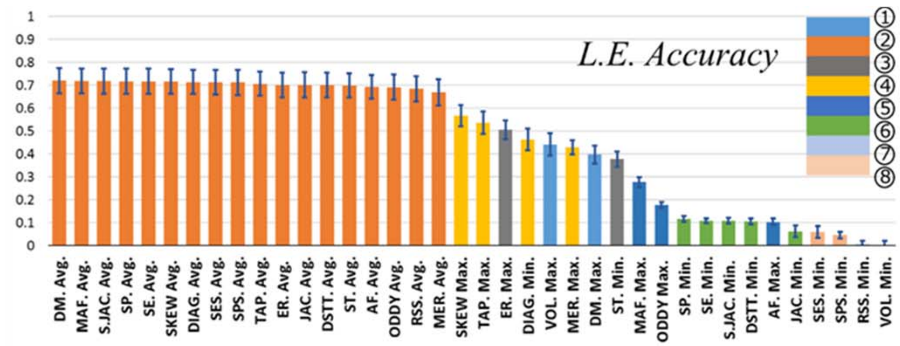
Correlations between individual quality metrics for hex-meshes and the simulation quality

Average correlation matrix

Application-Dependent Metric Correlation

Which metric(s) are more relevant to the simulation quality?

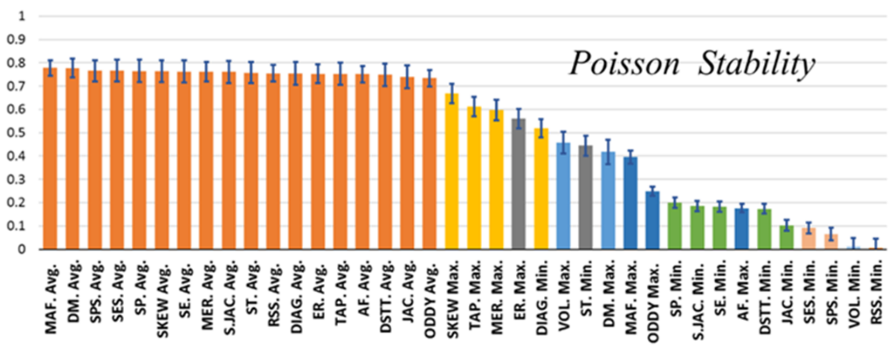


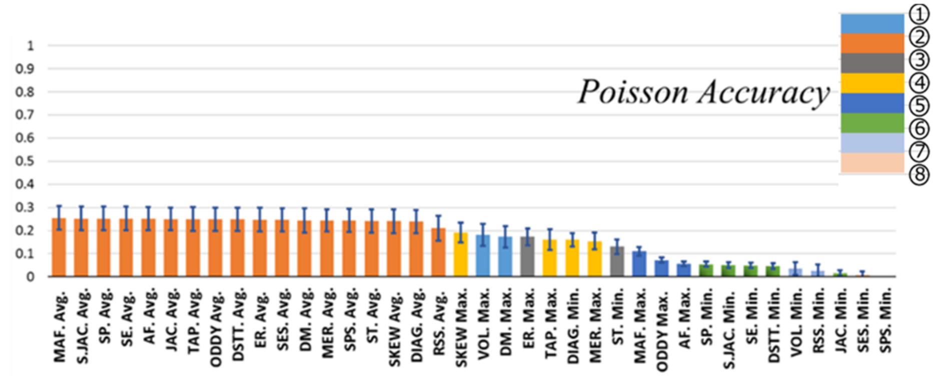


For the linear elasticity problem

Application-Dependent Metric Correlation

Which metric(s) are more relevant to the simulation quality?

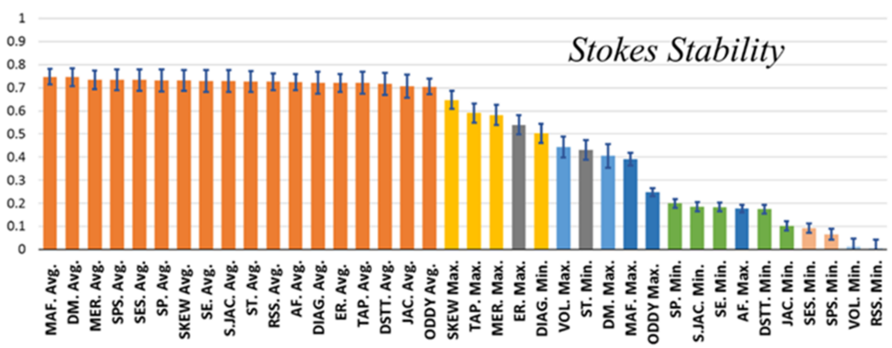


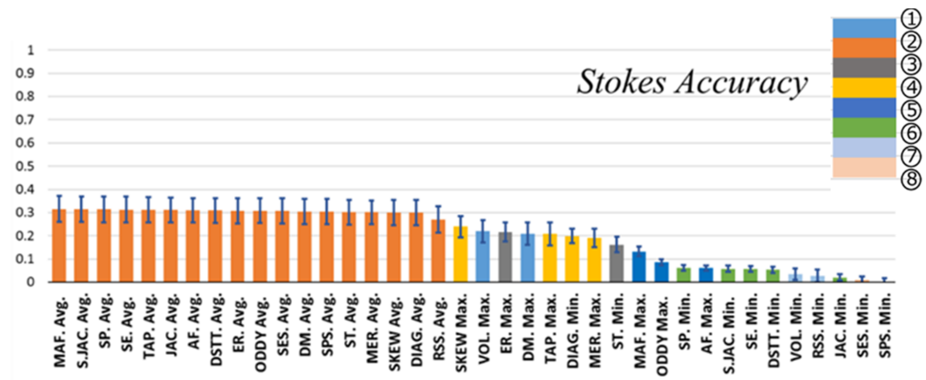


For the Poisson's equation solving

Application-Dependent Metric Correlation

Which metric(s) are more relevant to the simulation quality?

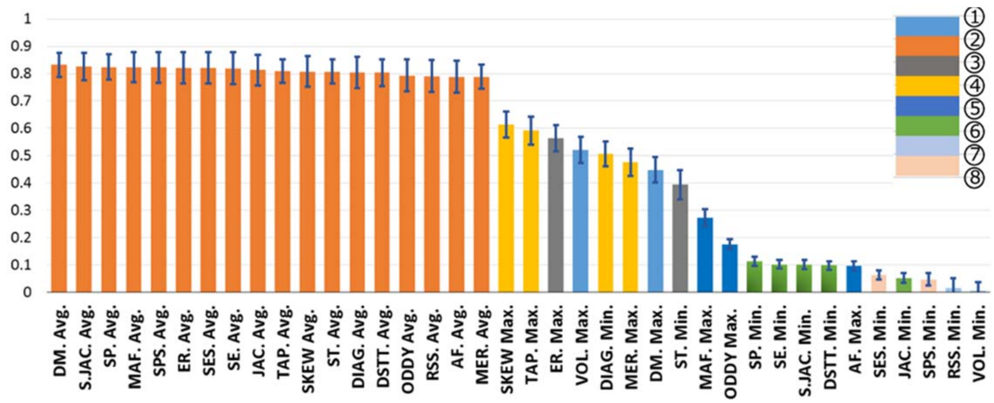




For the Stokes equation solving

Application-Dependent Metric Correlation

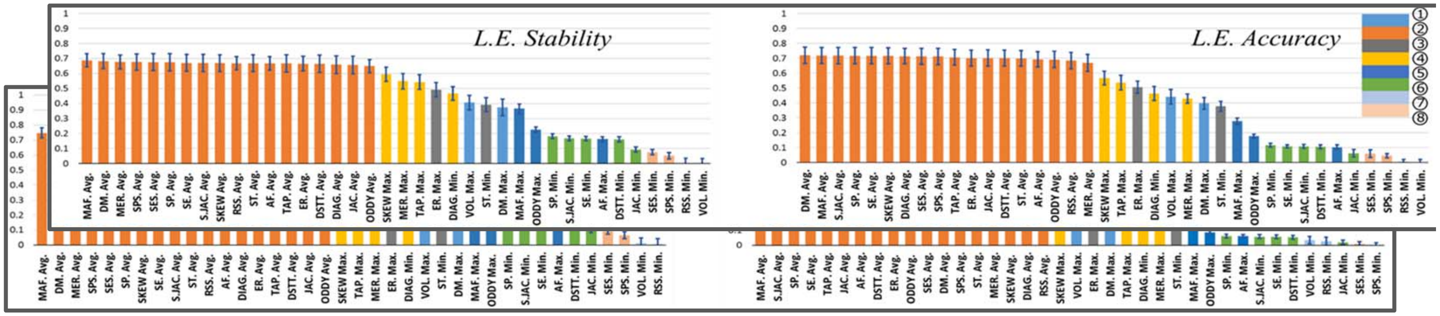
Which metric(s) are more relevant to the simulation quality?



For a Poisson's boundary problem with known solution

Application-Dependent Metric Correlation

Observations



- All average metrics have stronger correlations with both the Accuracy and Stability than those minimum/maximum metrics.
- Among those metrics, MAF. Avg. has the strongest correlation with the Stability for all three applications and DM. Avg. ranked second. For Accuracy, while MAF. Avg. has the strongest correlation with the Poisson’s and Stokes simulations, DM. Avg. tops the others for the linear elasticity simulations
- The correlations of the metrics (especially average metrics) with the Stability are much stronger than their correlations with the Accuracy in the Poisson’s and Stokes equation solving applications
- SKEW Max. that measures the orthogonality of the principal axes of a hexahedron, has the highest ranking among all Min/Max metrics for most applications, **while the well-known S. Jac. Min is ranked much lower in all experiments**.

Application-Dependent Metric Correlation

Recommendations

- To achieve accurate and stable computations in solving isotropic elliptic PDEs, the average metrics should be the focus to improve once the mesh is inversion-free;
- Metrics that characterize the conditioning of the elements, e.g. MAF. Avg., DM. Avg., SKEW Max., MER. Max. and TAP. Max., have stronger correlations with the quality of solving elliptic PDEs than other metrics, and thus, should be the quality that the meshing techniques try to optimize.

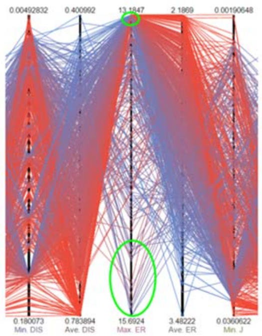
Use it with your own risk! :-)

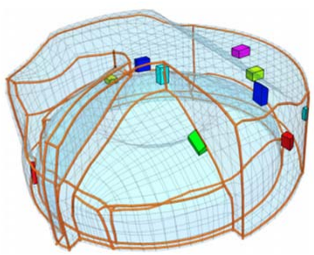
Conclusion

- We propose a practical yet efficient framework for the evaluating available quality metrics for hex-meshes;
- Metrics are classified into a set of groups, and the number of metrics has been reduced;
- W.r.t. three applications, we have identified the most correlated metrics;
- Online access for all the dataset and source code.

Limitations and Future Work

- The result and conclusion may be influenced by the purpose of testing (or, the dataset generation strategy);
- A better dataset sampling on various metrics that have non-orthogonal relationships is highly needed;
- Other degree of freedoms, e.g., the location of the worst element, connectivities, discretization, and model types may matter.





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- Thank all the anonymous reviewers for their valuable comments and suggestions.
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Thank you for
your attention!