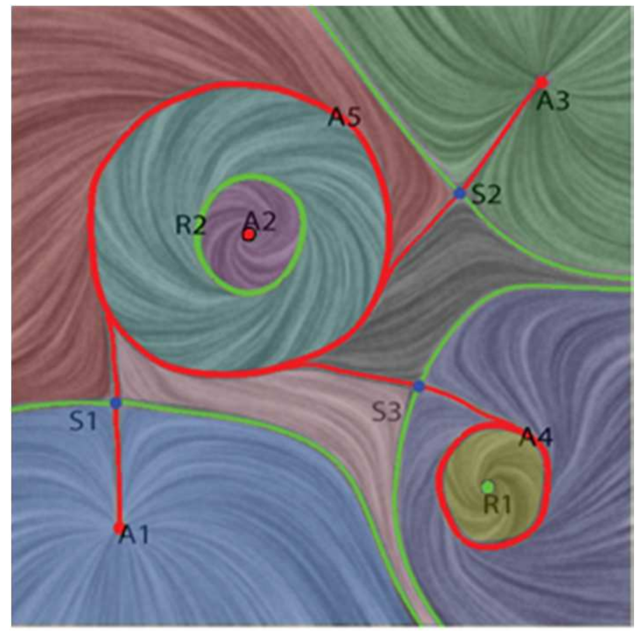
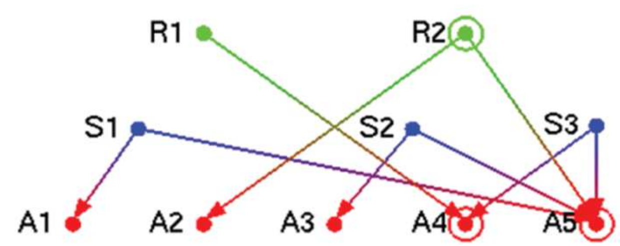
Discrete Vector Field Topology – Morse Decomposition

Differential Vector Field Topology

- Vector field topology provides qualitative (structural) information of the underlying dynamics
- It usually consists of certain critical features and their connectivity, which can be expressed as a graph, e.g. vector field skeleton [Helman and Hesselink 1989]
  - Fixed points
  - Periodic orbits
  - Separatrices



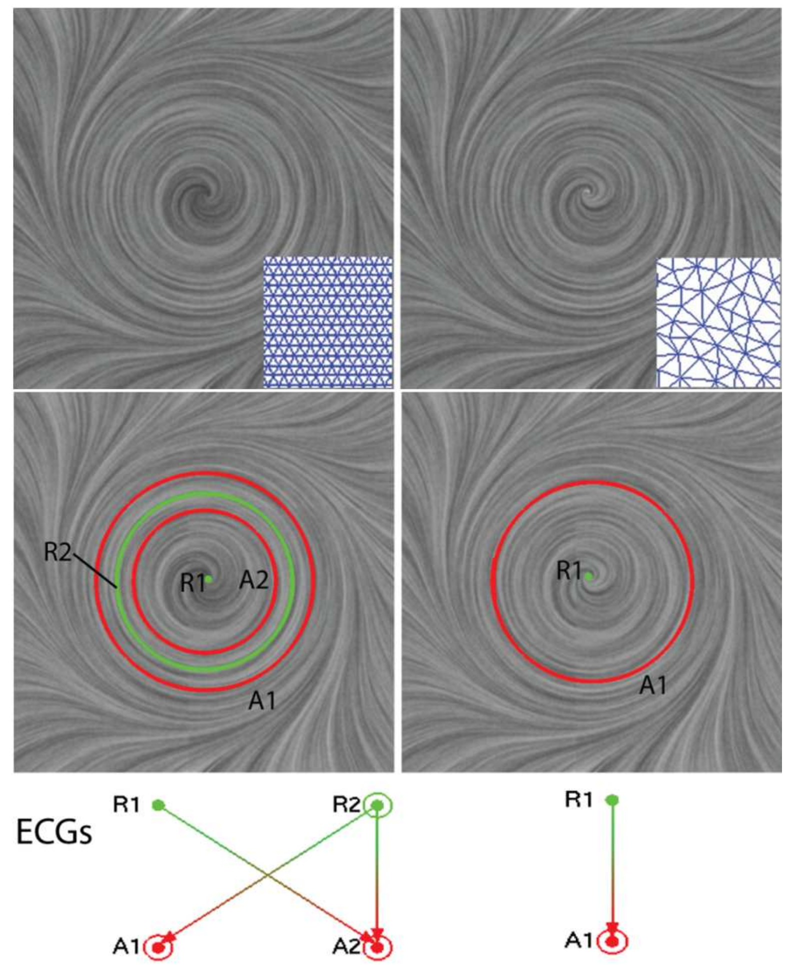


What is the problem?

Instability of Differential Topology (1)

Case 1: different sampling

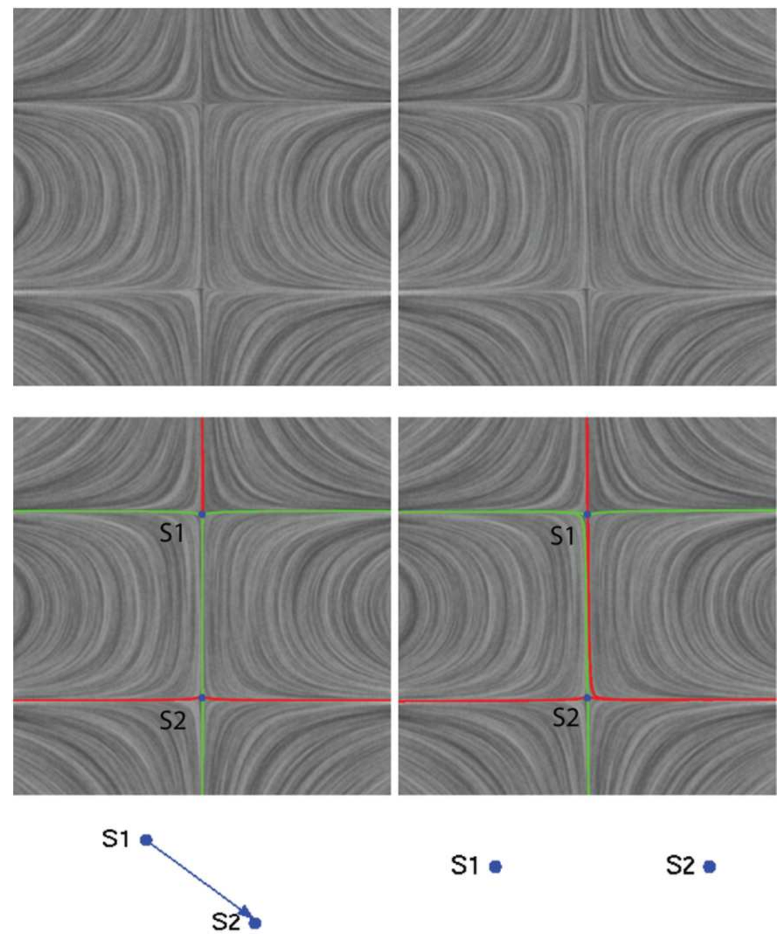
$$V(r,\theta) = \begin{pmatrix} r(r-1)(k-(r-1)^2) \\ 1 \end{pmatrix}$$



Instability of Differential Topology (2)

Case 2: noise in the data

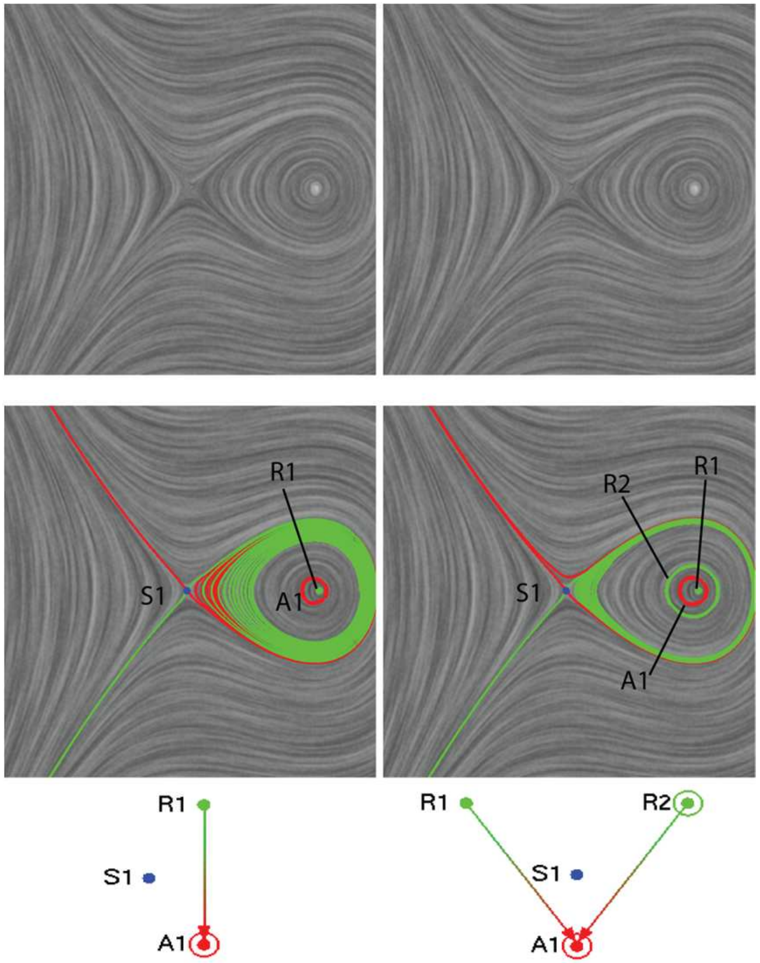
$$V(x,y)=\begin{pmatrix} y \\ kx - x(1-x) \end{pmatrix}$$



Instability of Differential Topology (3)

Case 3: different numerical integration schemes

$$V(x,y)=\begin{pmatrix} -xy \\ (y^2-1) \end{pmatrix}$$

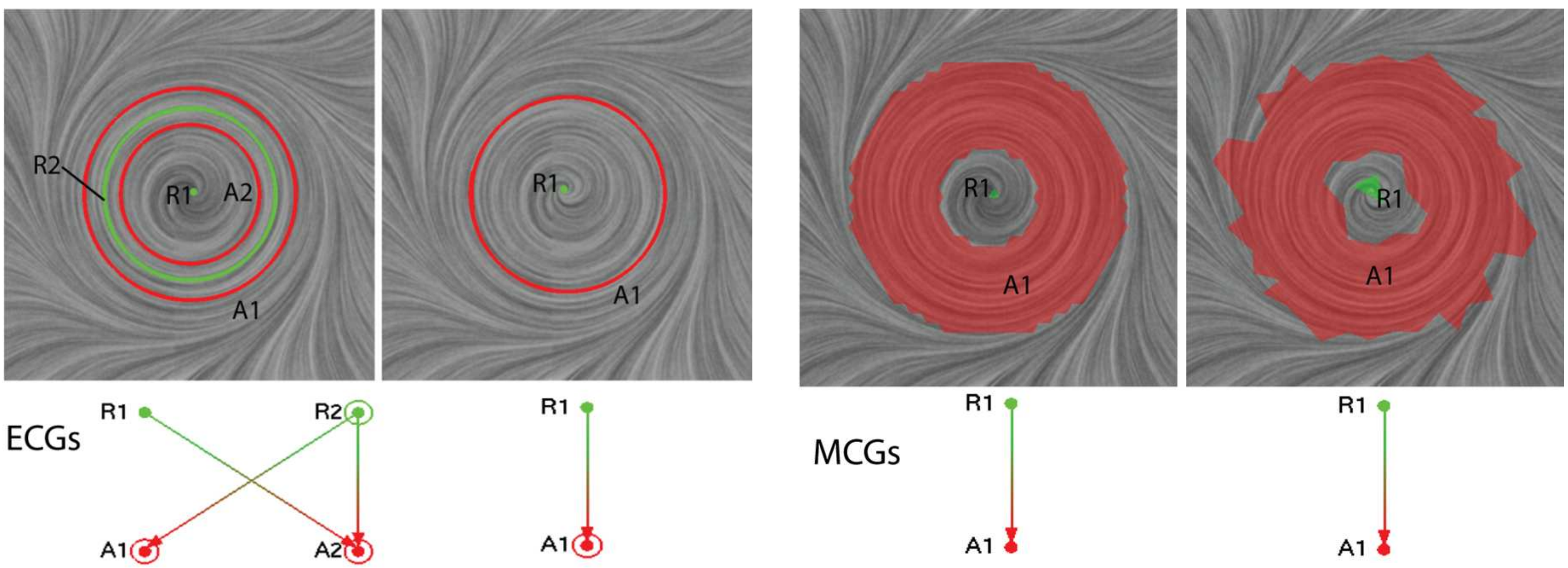


2D Vector Field Topology

- Differential topology
  - Topological skeleton [Helman and Hesselink 1989; CGA91]
  - Entity connection graph [Chen et al. TVCG07]
- Discrete topology
  - Morse decomposition [Conley 78] [Chen et al. TVCG08, TVCG11a]
  - PC Morse decomposition [Szymczak EuroVis11] [Szymaczak and Zhang TVCG12][Szymaczak TVCG12]
- Combinatorial topology
  - Combinatorial vector field [Forman 98]
  - Combinatorial 2D vector field topology [Reininghaus et al. TopoInVis09, TVCG11]

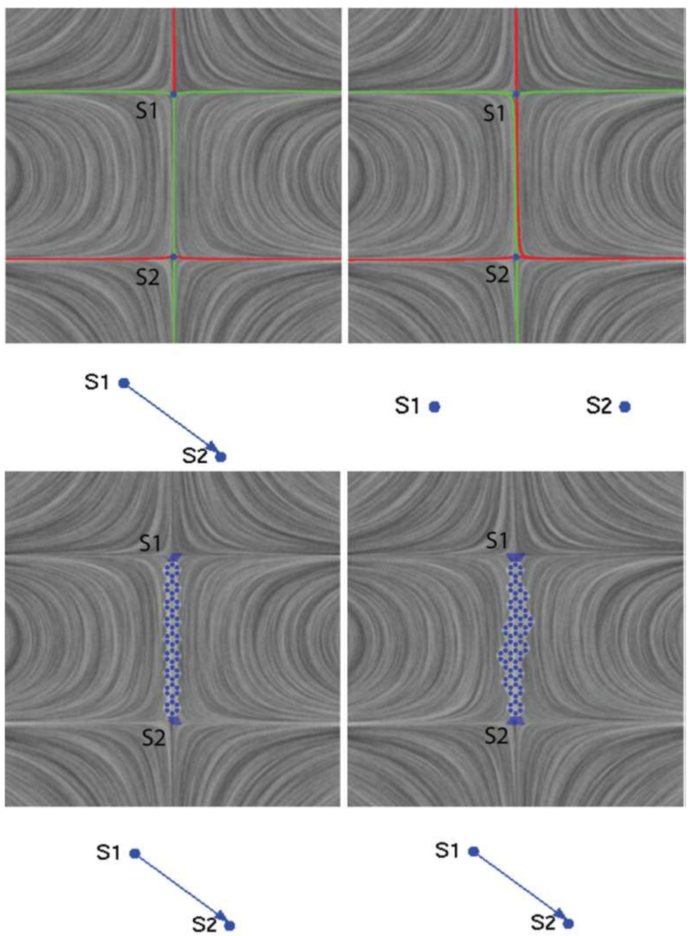
Morse Decomposition Results

- Stable



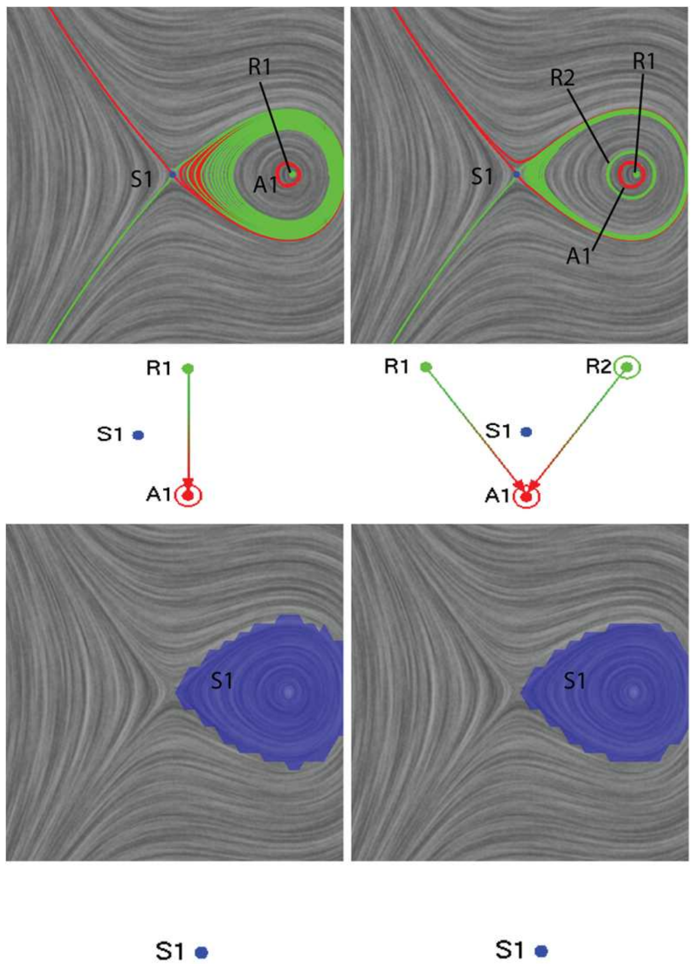
Morse Decomposition Results

- Stable



Morse Decomposition Results

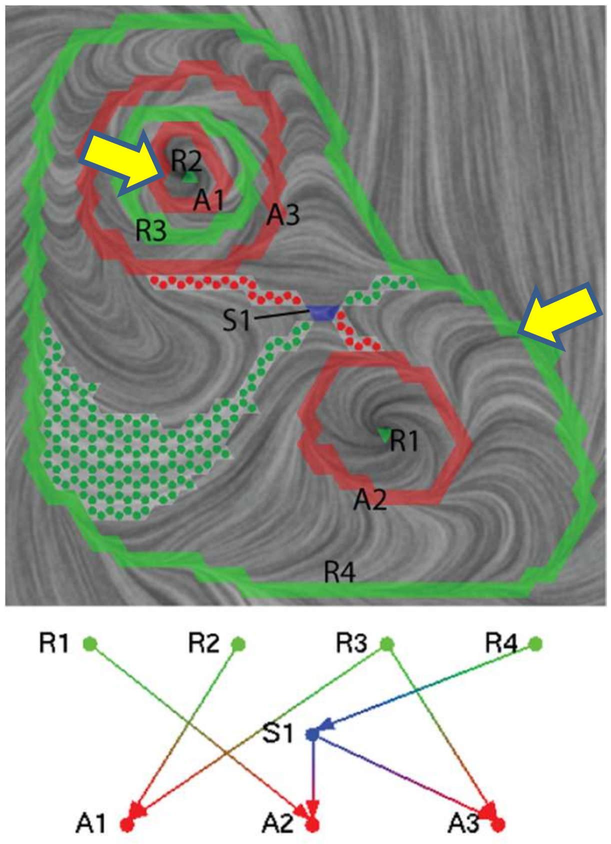
- Stable



What is a Morse Decomposition?

- *A* **Morse decomposition** of surface ***X*** for the flow φ is a finite collection of disjoint compact invariant sets, called **Morse sets**.

- The result of a Morse decomposition computation is a directed graph called **Morse connection graph (MCG)**.

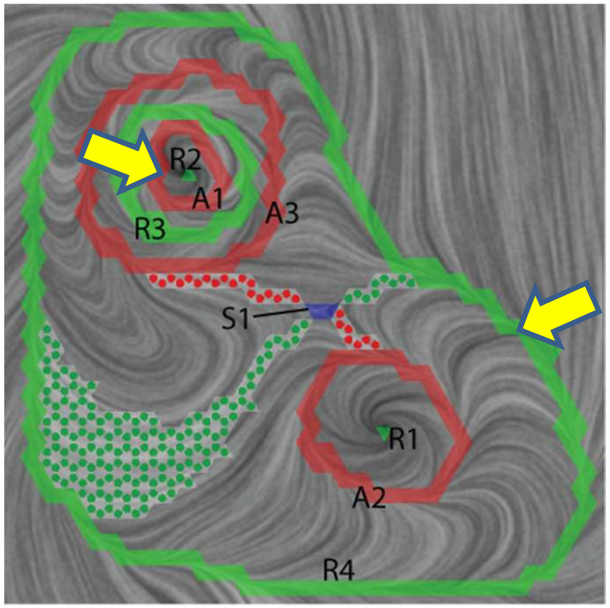


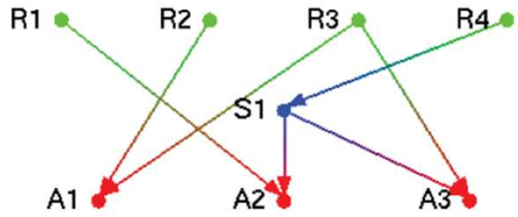
MCG Definition

- An **MCG**

$$M(X,\varphi,P,>)=\{M(p)|p\in(P,>)\}$$

  - is an acyclic directed graph, whose nodes P are Morse sets, the set of directed edges is a strict partial order $>$
  - such that for any $x\notin \cup_{p\in P} M(p)$, there exist $p>q$ in P and $\alpha(x)\subset M(p)$ and $\omega(x)\subset M(q)$

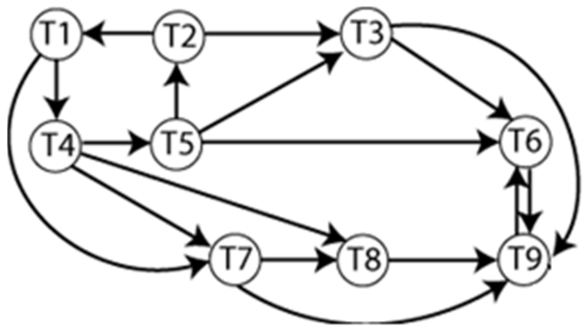




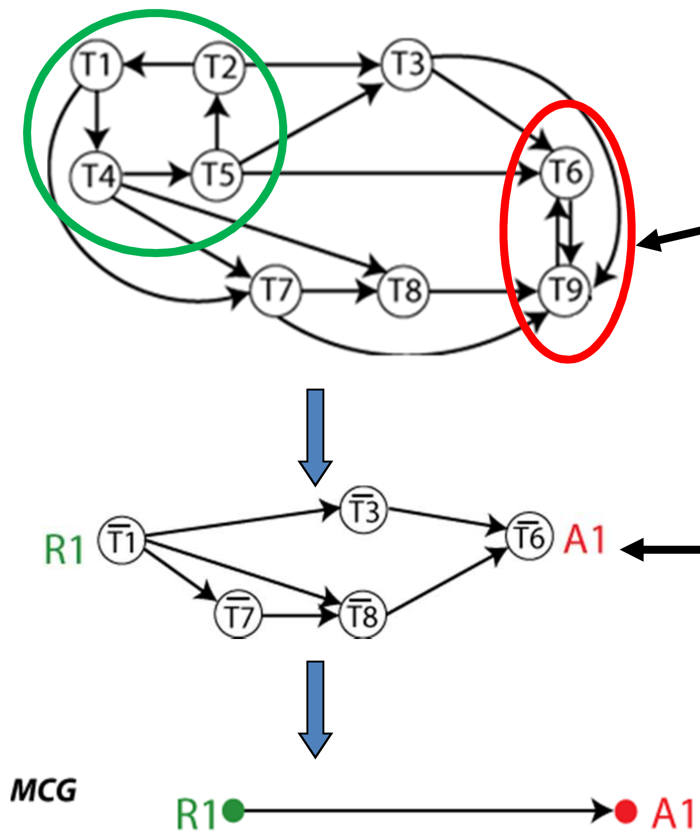
A Pipeline of Morse Decomposition

Vector field on a triangulation

Flow combinatorialization



A Pipeline of Morse Decomposition



Vector field on a triangulation

Flow combinatorialization

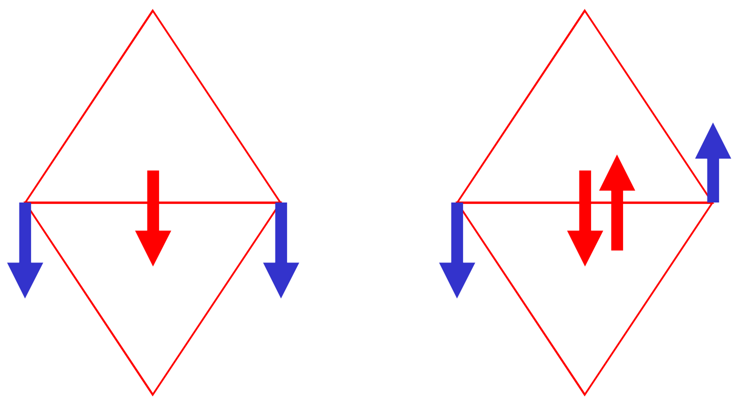
Strongly connected component extracting

Constructing a quotient graph

Computing MCG

Flow combinatorialization

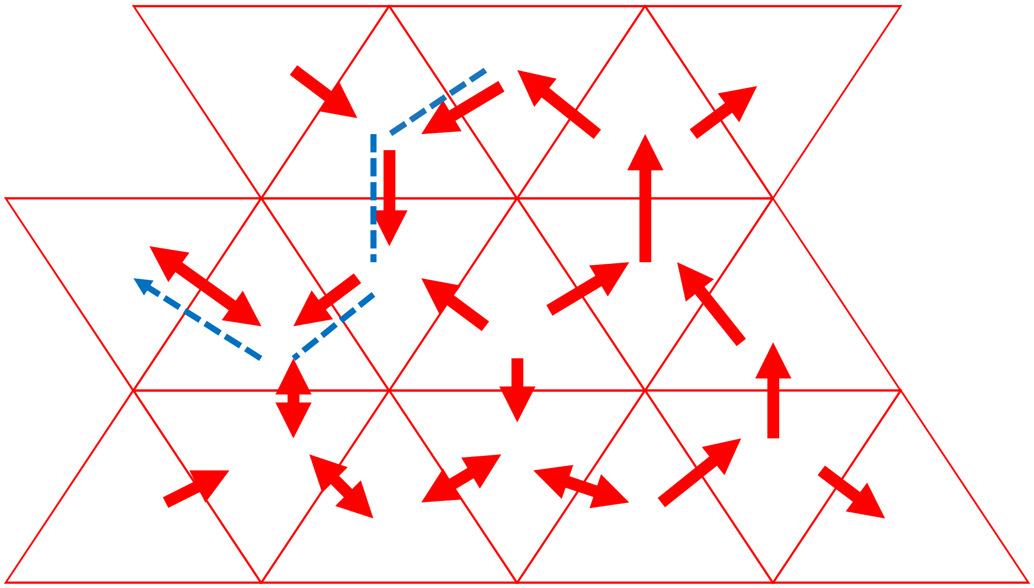
- A geometry-based method



(1) (2)

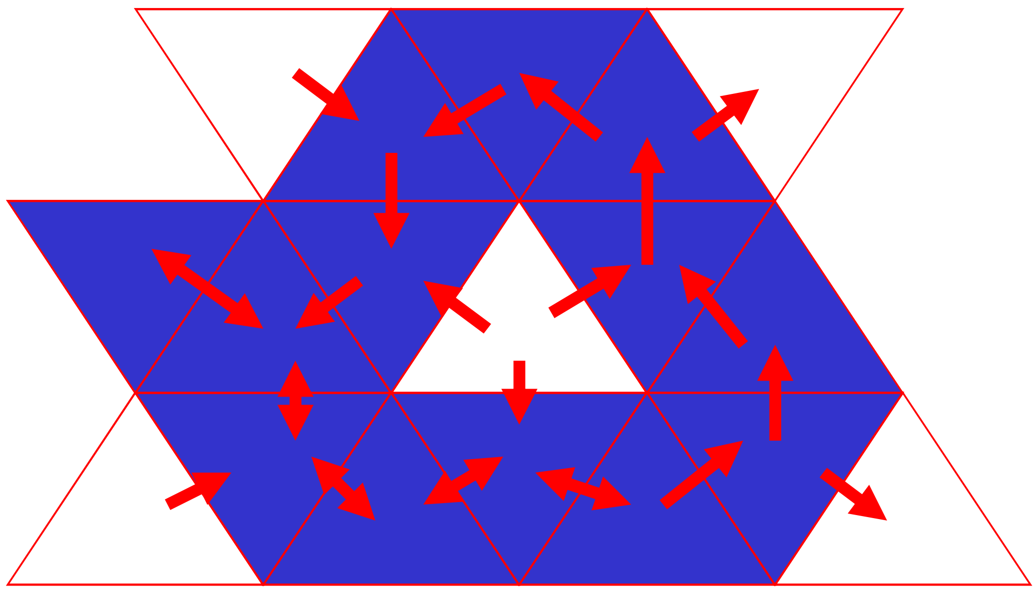
Flow combinatorialization

- Flow combinatorialization encodes flow dynamics in a directed graph



Extract Region of Recurrence

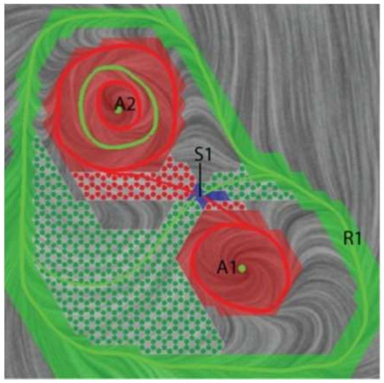
- Regions of recurrent flow correspond to the strongly-connected components of the directed graph!



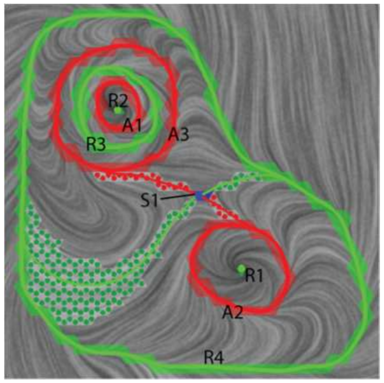
Properties of Morse Decomposition

- Morse decomposition is a family of Morse sets, i.e. disjoint compact sets such that:
  1) Any trajectory that is NOT contained in the union of Morse sets connect two different Morse sets
  2) No cycle exist in the 'is connect to' relation on the family of Morse sets

- Properties
  1) Flow gradient like outside Morse sets
  2) Morse sets capture all recurrent dynamics
  3) Not unique (determined by a *parameter* τ)
     - Coarse: more stable
     - Fine: easier to understand
     - Support of multi-scale analysis

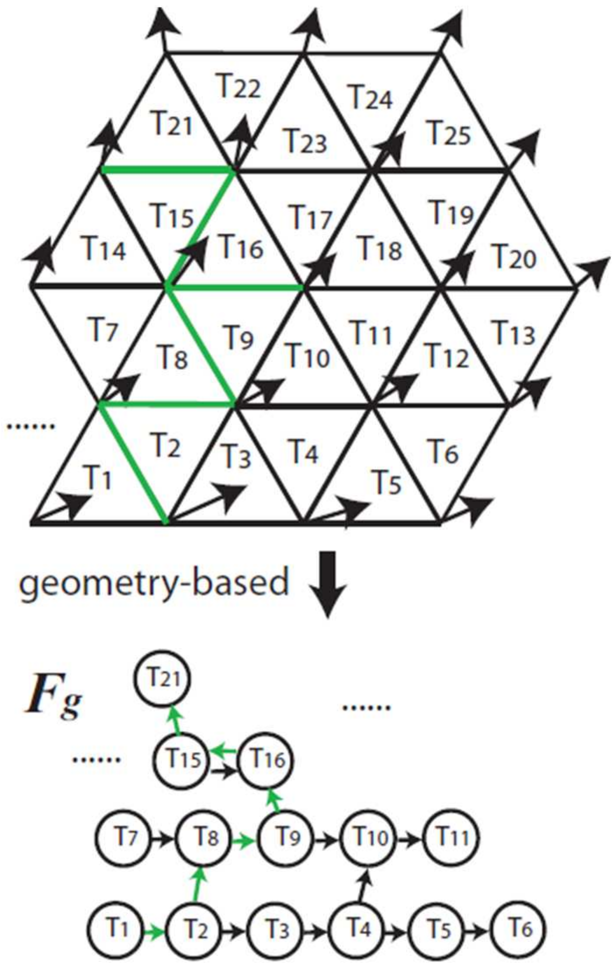


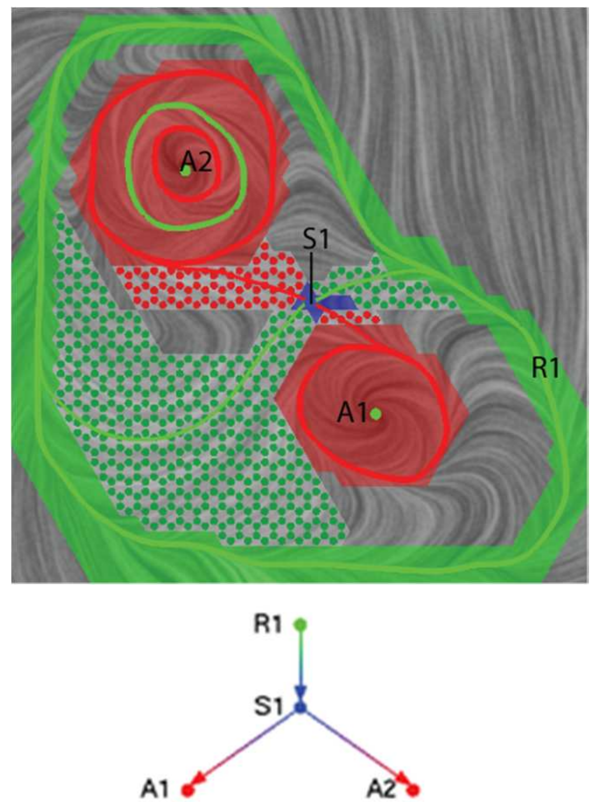
coarse



fine

Issue of Geometry Based Flow Combinatorialization

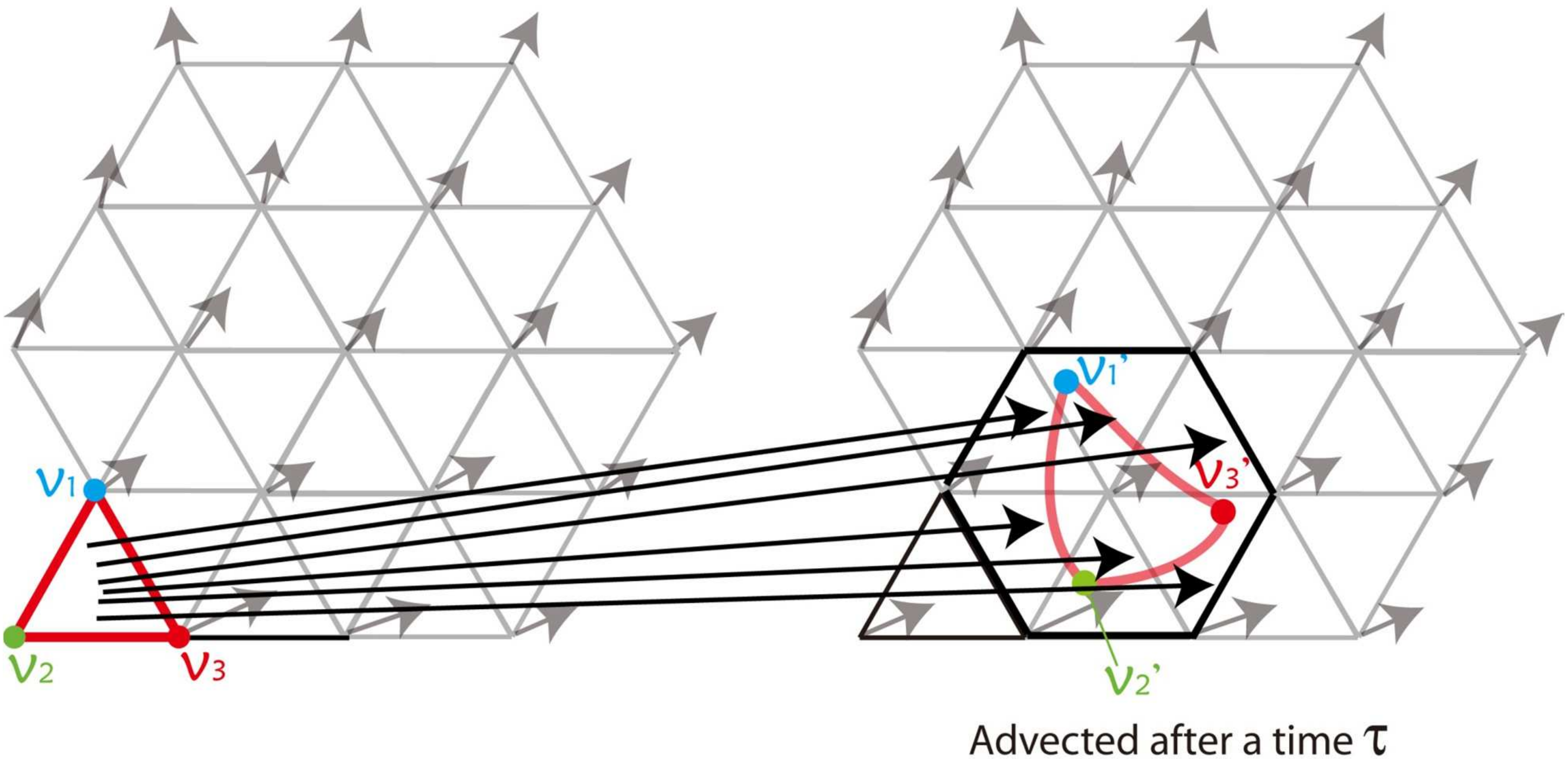




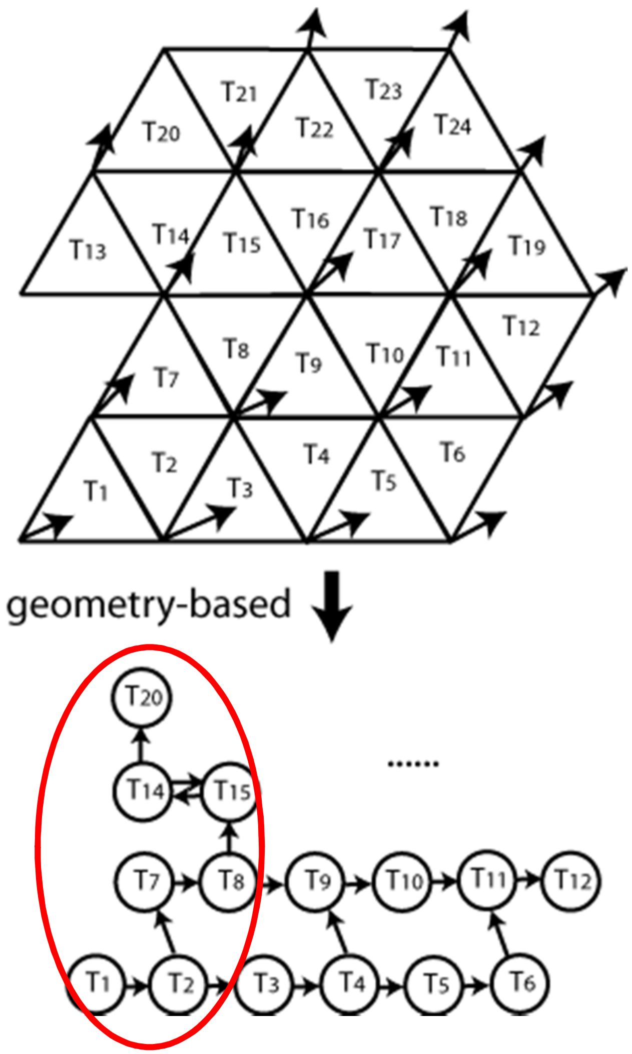
How to achieve finer decomposition?

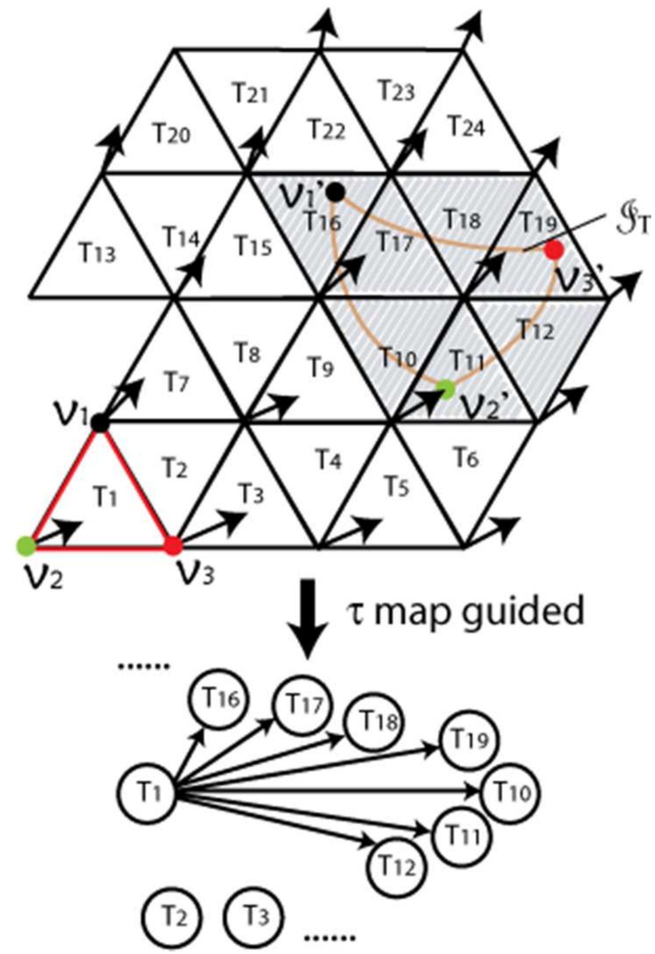
Flow Combinatorialization

$\boldsymbol{\tau}$-map based method

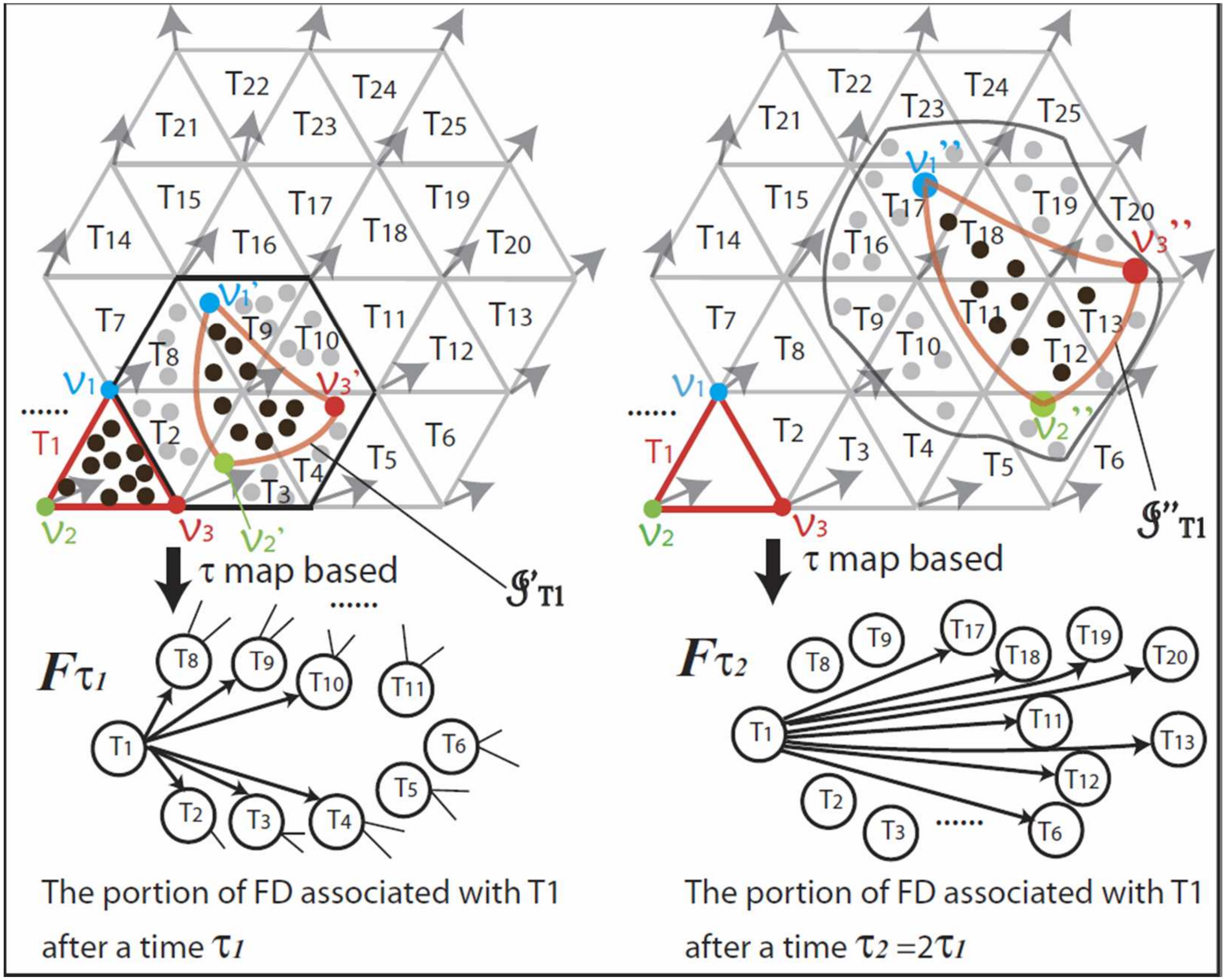


Why τ-map based method is better

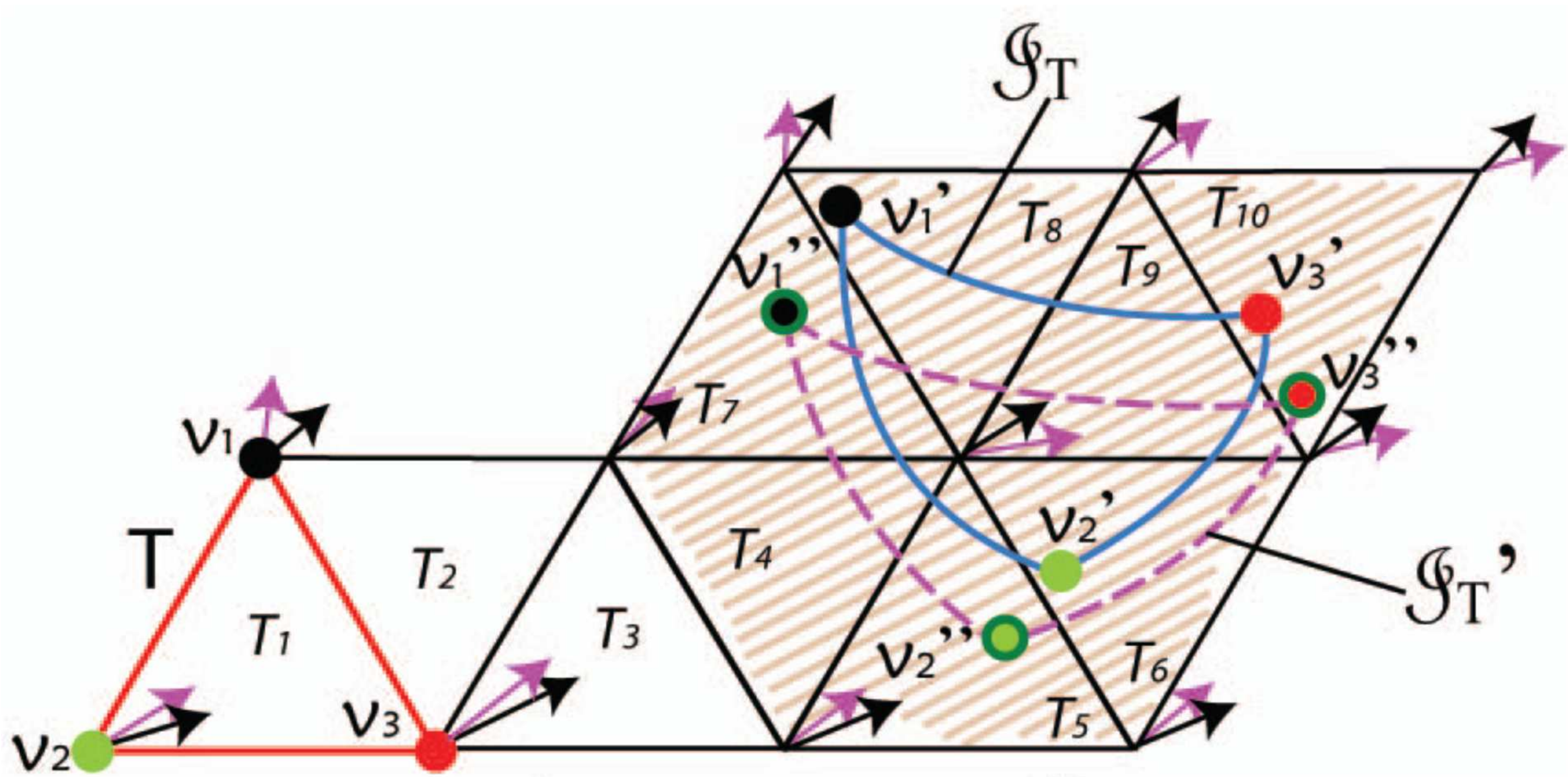




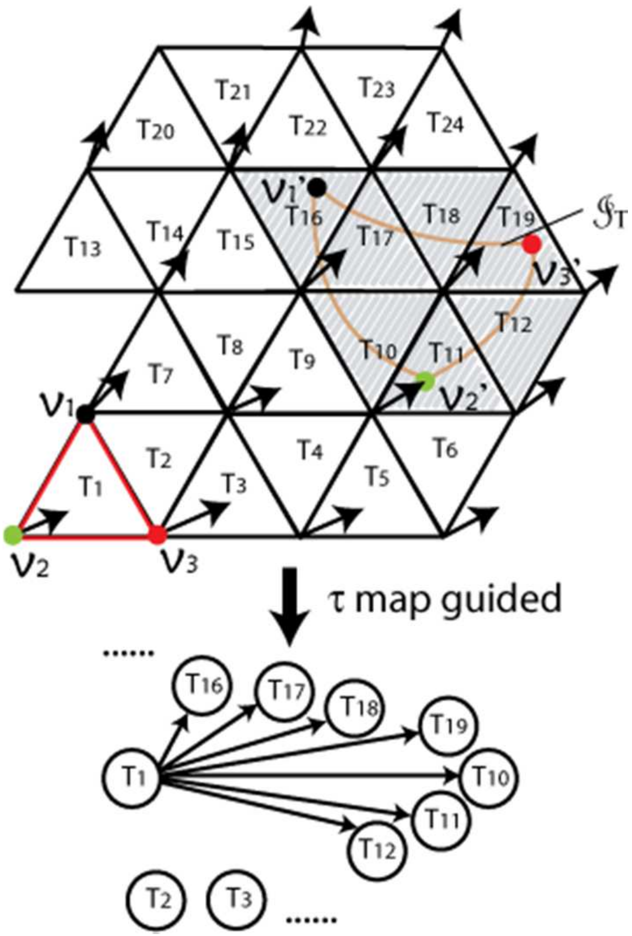
Why the larger the τ is the finer the decomposition?

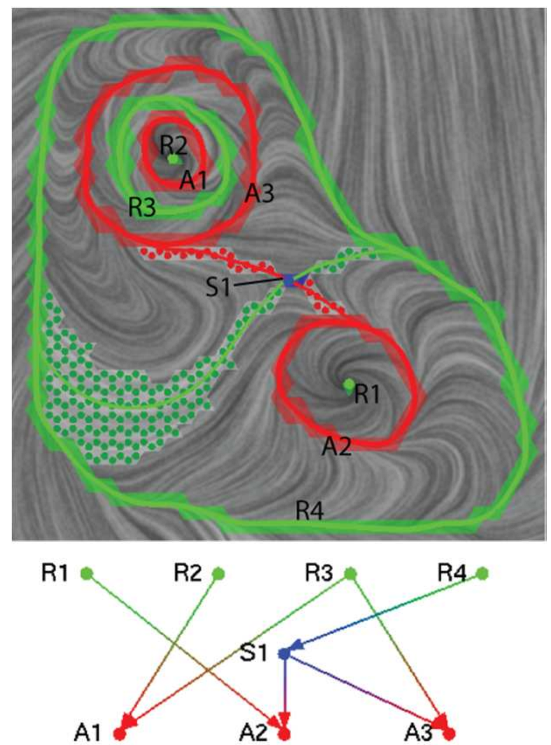


Why Is MCG Stable?



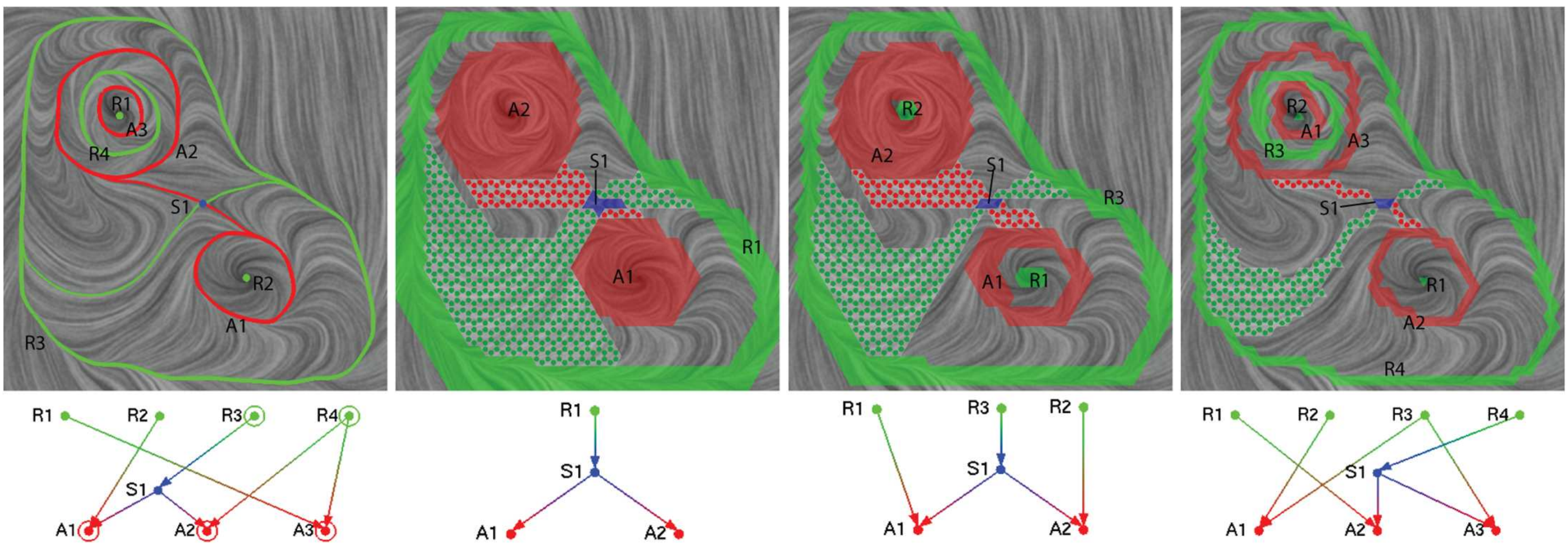
τ-Map Based Flow Combinatorialization Result





Morse Decomposition is **Not Unique**

They are all correct!



Small τ *Large τ*

ECG

MCGs with increasing τ

In order to construct the MCG and visualize it, we need to classify the Morse sets and place them in the proper layers as ECG does

Can Poincaré Indices be Used Here?

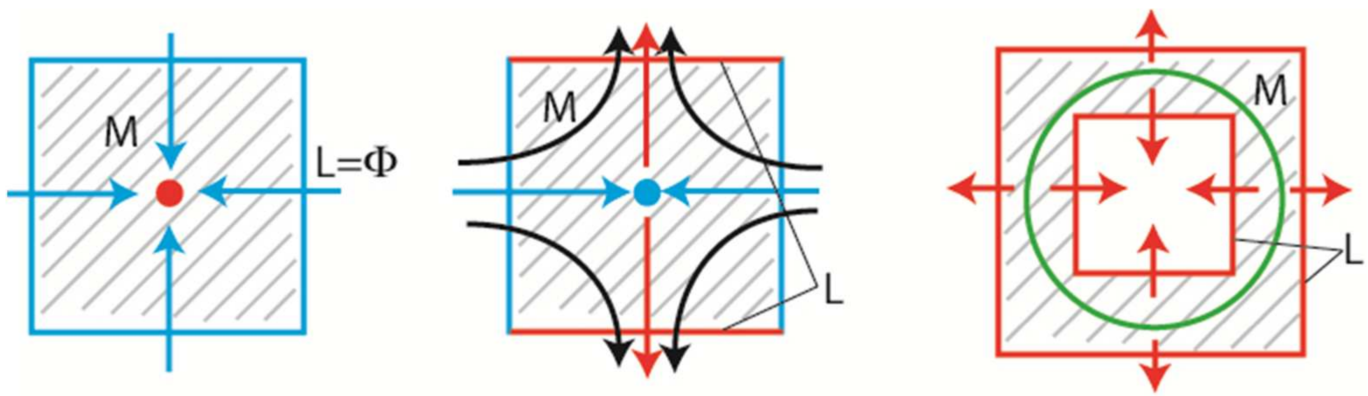
- Consider an **isolated** fixed point x_0, there is a neighborhood N enclosing x_0 such that there are no other fixed points in N or on the boundary curve ∂N
  - if $\mathbf{I}(\partial N, V) = 1$, x_0 is either a source or a sink;
  - if $\mathbf{I}(\partial N, V) = -1$, x_0 is a saddle.

- The Poincarè index of a fixed point free region is 0

- Poincaré Index of a periodic orbit is zero as well!!!

Conley Index

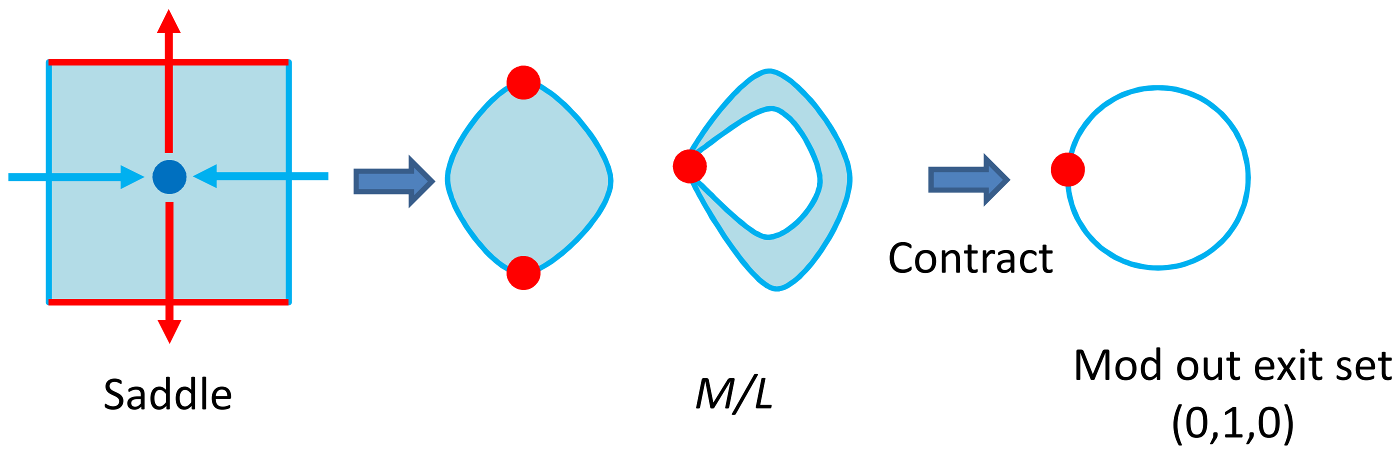
- There is an index, called **Conley index** that we use to classify Morse sets.
- For an isolating block M, its Conley index is the homotopy type of the quotient space M/L where L is the *exit set* (the subset of the boundary of M consisting of all exit points).



(a) (b) (c)

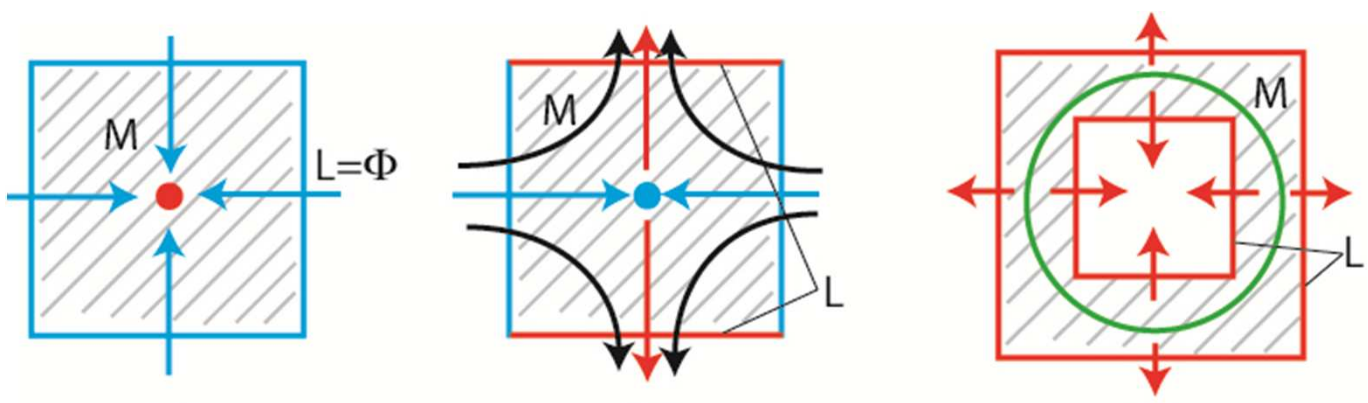
Conley Index

- For an isolating block *M*, its Conley index is computed as the Betti numbers (β_0, β_1, β_2) of the *quotient space M/L* where *L* is the *exit set* (the subset of the boundary of *M* consisting of all exit points).



Conley Index Computation (β_0)

- β_0 simply counts the number of connected components in M that **do not** attach with L.
- Since M is always connected in our cases, then β_0 is zero if $L \neq \Phi$ and 1 otherwise.



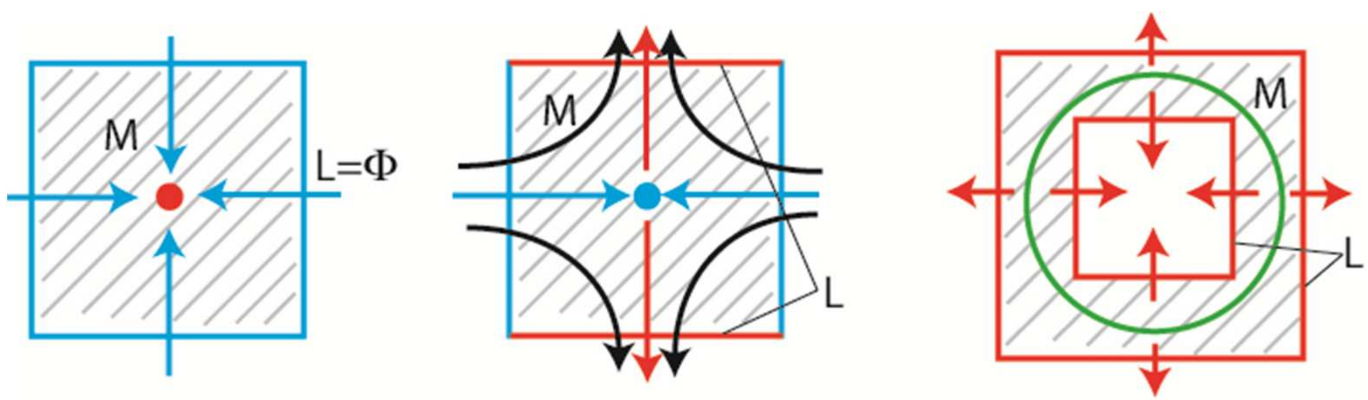
(a) $\beta_0 = 1$

(b) $\beta_0 = 0$

(c) $\beta_0 = 0$

Conley Index Computation (β_2)

- β_2 is equal to the number of connected components of M whose entire boundary is contained in L.
- Since M is connected, then $\beta_2 = 1$ if all boundaries are contained in L.



(a) (b) (c)

$\beta_2 = 0$ $\beta_2 = 0$ $\beta_2 = 1$

Conley Index Computation (β_1)

- Consider the Euler characteristic of M/L

$$X(M/L)=\beta_0-\beta_1+\beta_2$$

Also, $X(M/L) = X(M)-X(L)$

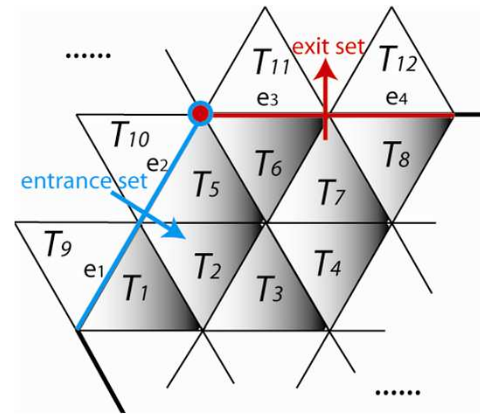
Thus, $\beta_1 = \beta_0+\beta_2 - [X(M) - X(L)]$

And, $X(L)=\beta_0(L)-\beta_1(L)+\beta_2(L)$

Where $\beta_0(L)$ is the number of the connected components in exit set L

$\beta_1(L)$ is the number of loops in exit set L

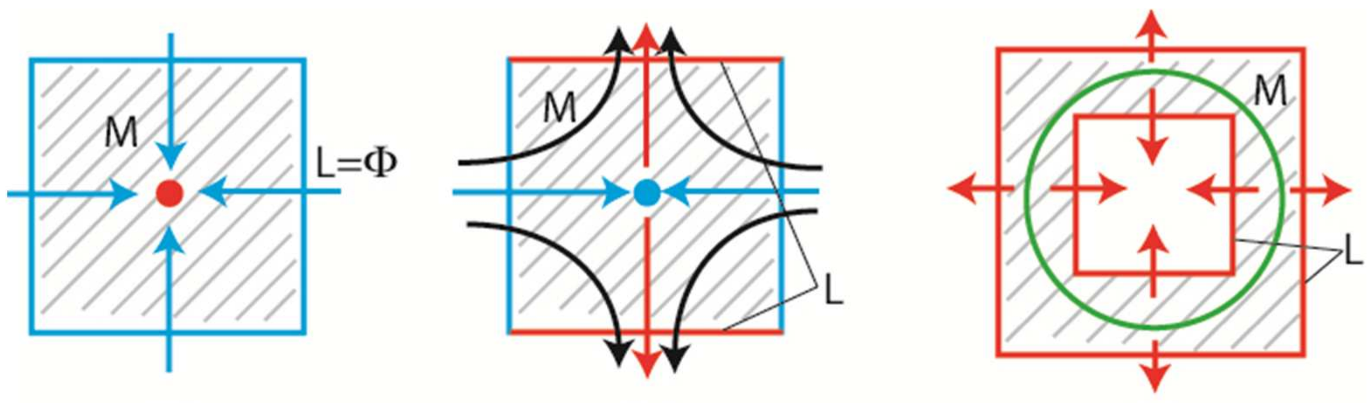
$\beta_2(L)$ is always zero for $1D$ curve



Conley Index Computation

Therefore,

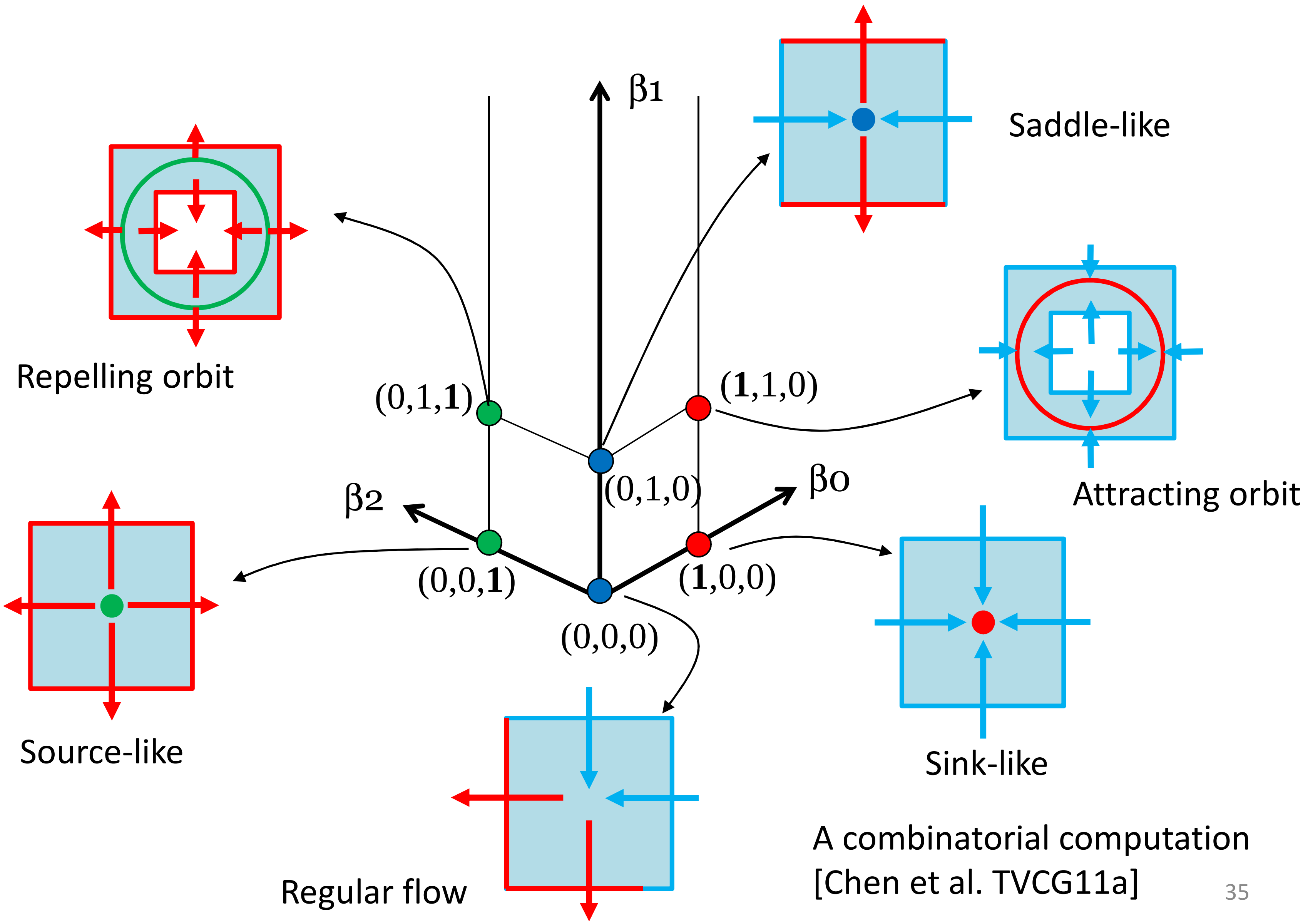
$$\beta_1 = \beta_0 + \beta_2 - [X(M) - X(L)]$$



(a) (b) (c)

(a)	(b)	(c)
$X(M) = 1$	$X(M) = 1$	$X(M) = 0$
$X(L) = 0$	$X(L) = 2$	$X(L) = 0$
$\beta_0 = 1$	$\beta_0 = 0$	$\beta_0 = 0$
$\beta_1 = 0$	$\beta_1 = 1$	$\beta_1 = 1$
$\beta_2 = 0$	$\beta_2 = 0$	$\beta_2 = 1$

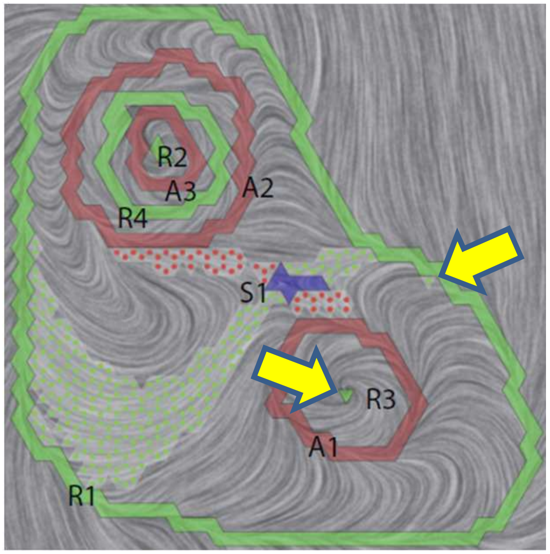
Basic Conley Indices in 2D



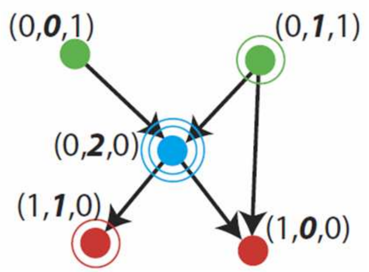
A combinatorial computation
[Chen et al. TVCG11a]

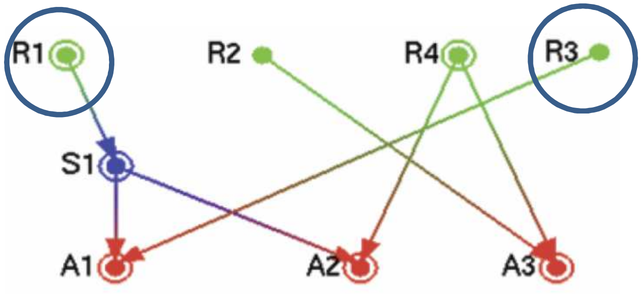
Classification of Morse Sets

- Note that β_0 and β_2 are at most 1, and CANNOT be both 1.
  - If $\beta_0 = 1$, **(A)ttractors**
  - If $\beta_2 = 1$, **(R)epellers**
  - Otherwise, **(S)addles**

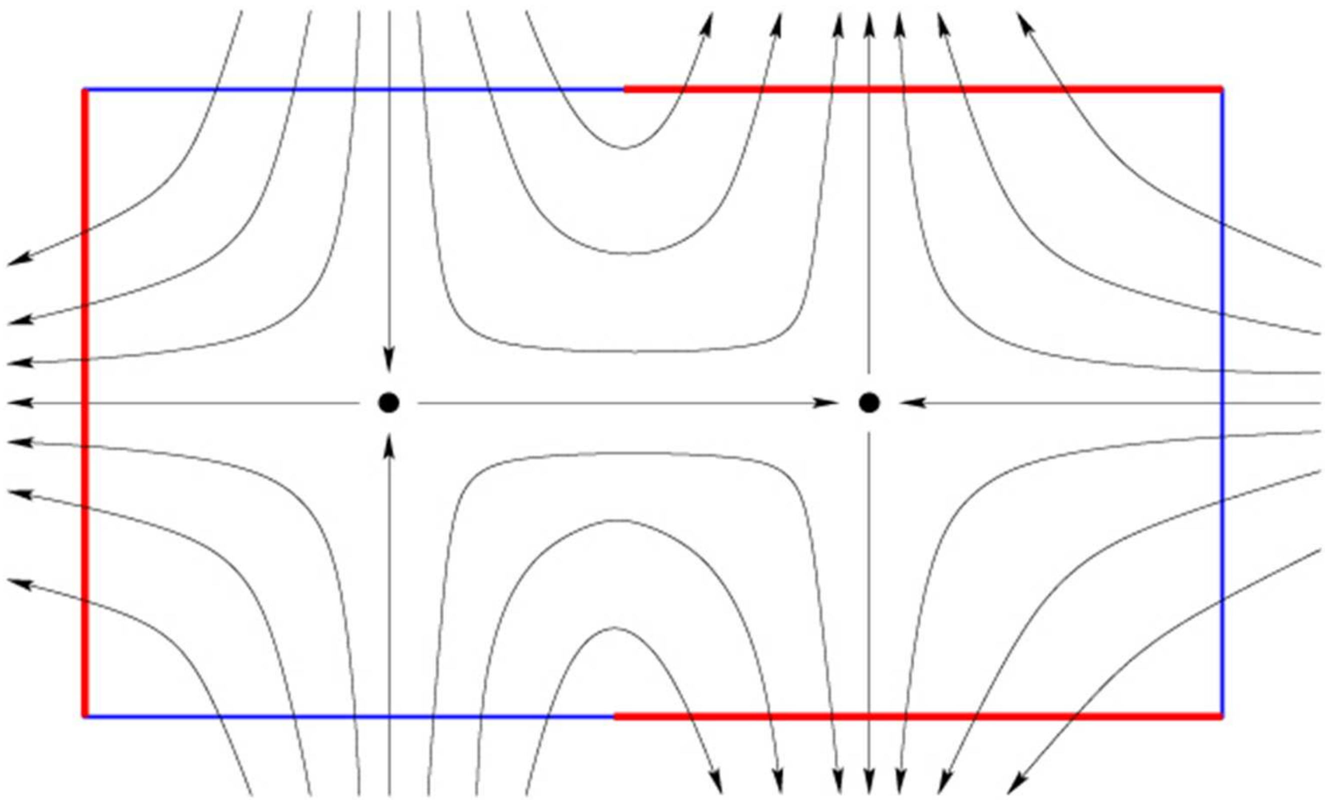


- Visualization



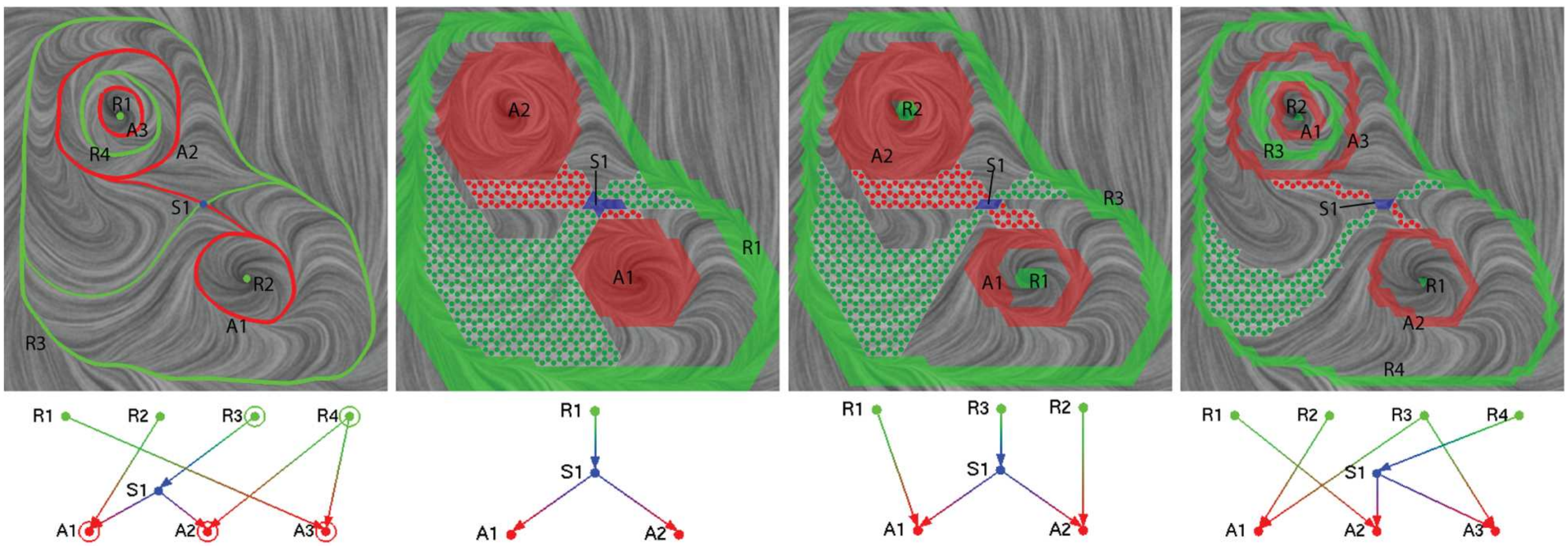


A More Complex Example



(0,2,0)

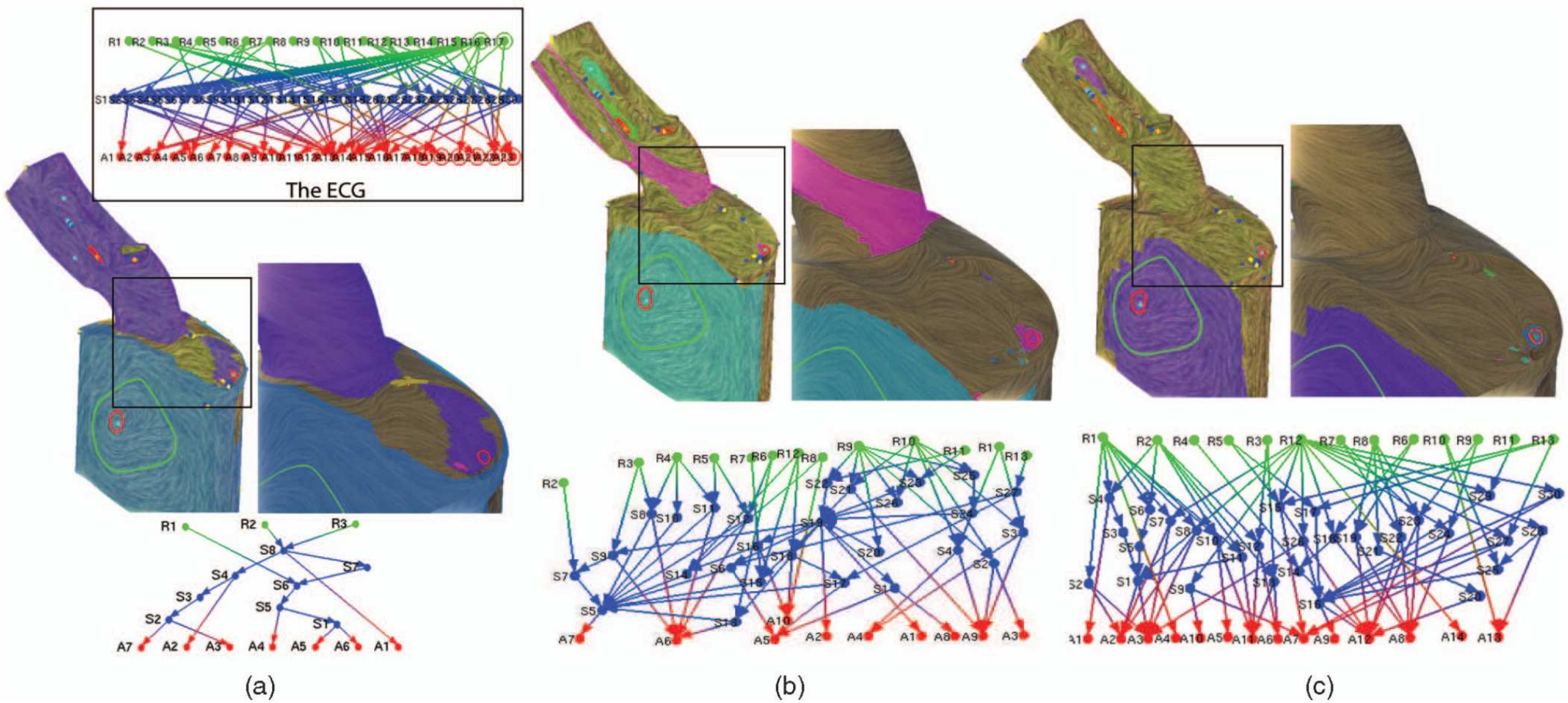
Results – Analytic Data



(a) ECG (b) Geometry-based (c) τ=6 (d) τ=24

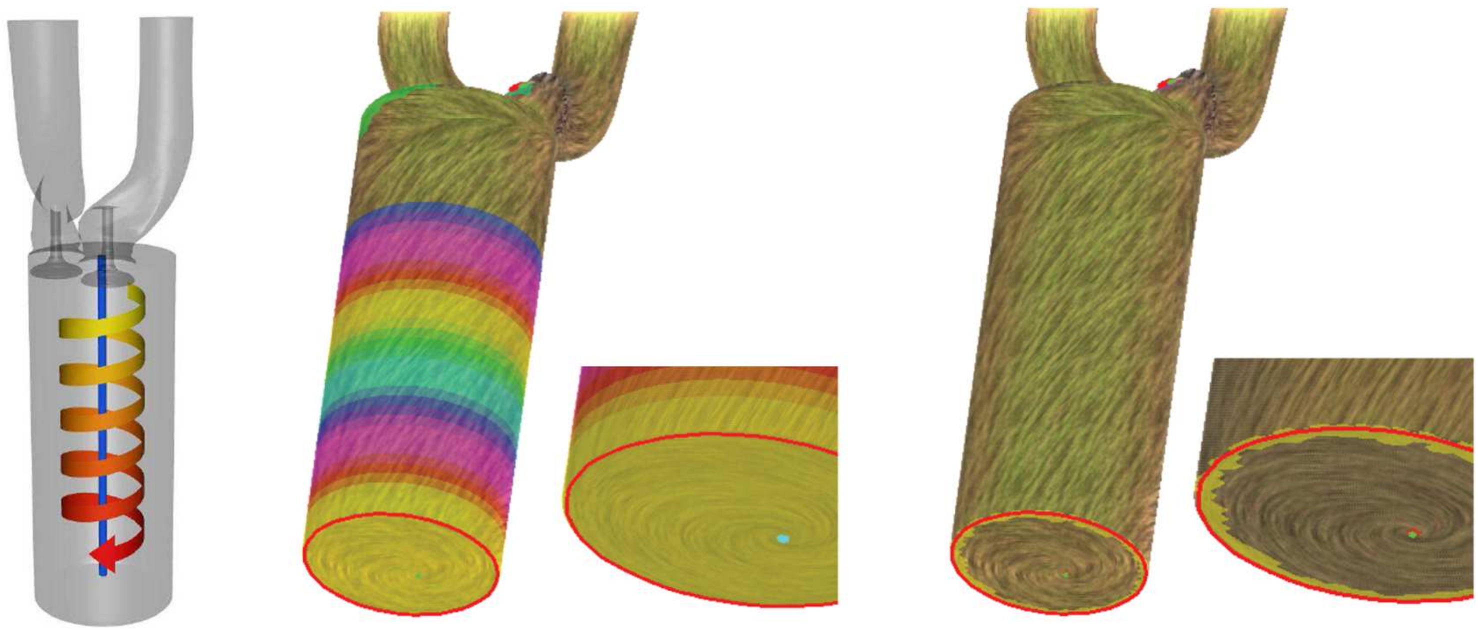
Results – Gas Engine

- Uniform τ



Results – Diesel Engine

- Uniform τ



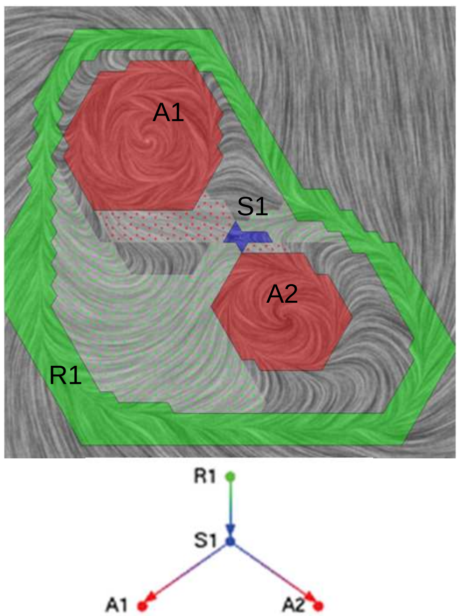
Results

- **Performance**

Dataset name	Number of triangles	Number of Morse sets	Time for flow combinatorialization (seconds)	Time for computing MCG (seconds)
Gas engine (t=0.1)	*26,298*	*50*	*27.8*	*7.9*
Gas engine (t=0.3)	*26,298*	*57*	*75.4*	*1.2*
Diesel engine (t=0.3)	*221,574*	*200*	*1101.3*	*37.7*

All the results are obtained in a 3.6 GHz PC with 3GB RAM.

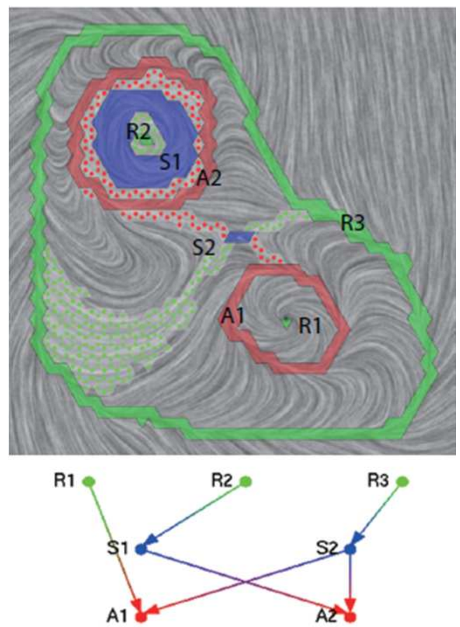
Issue (I) With Uniform τ



τ_1

time_1

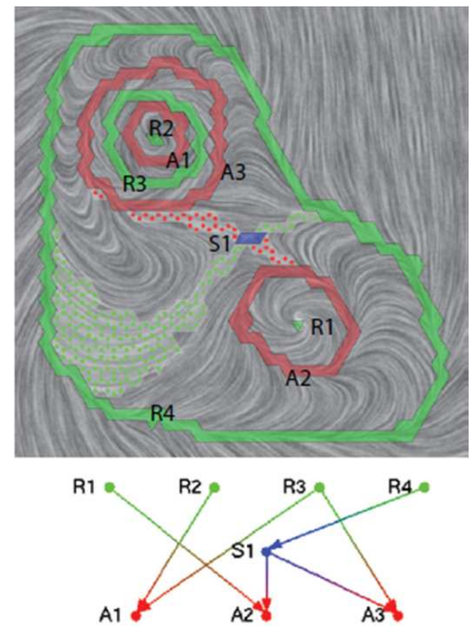
…



τ_i

time_i

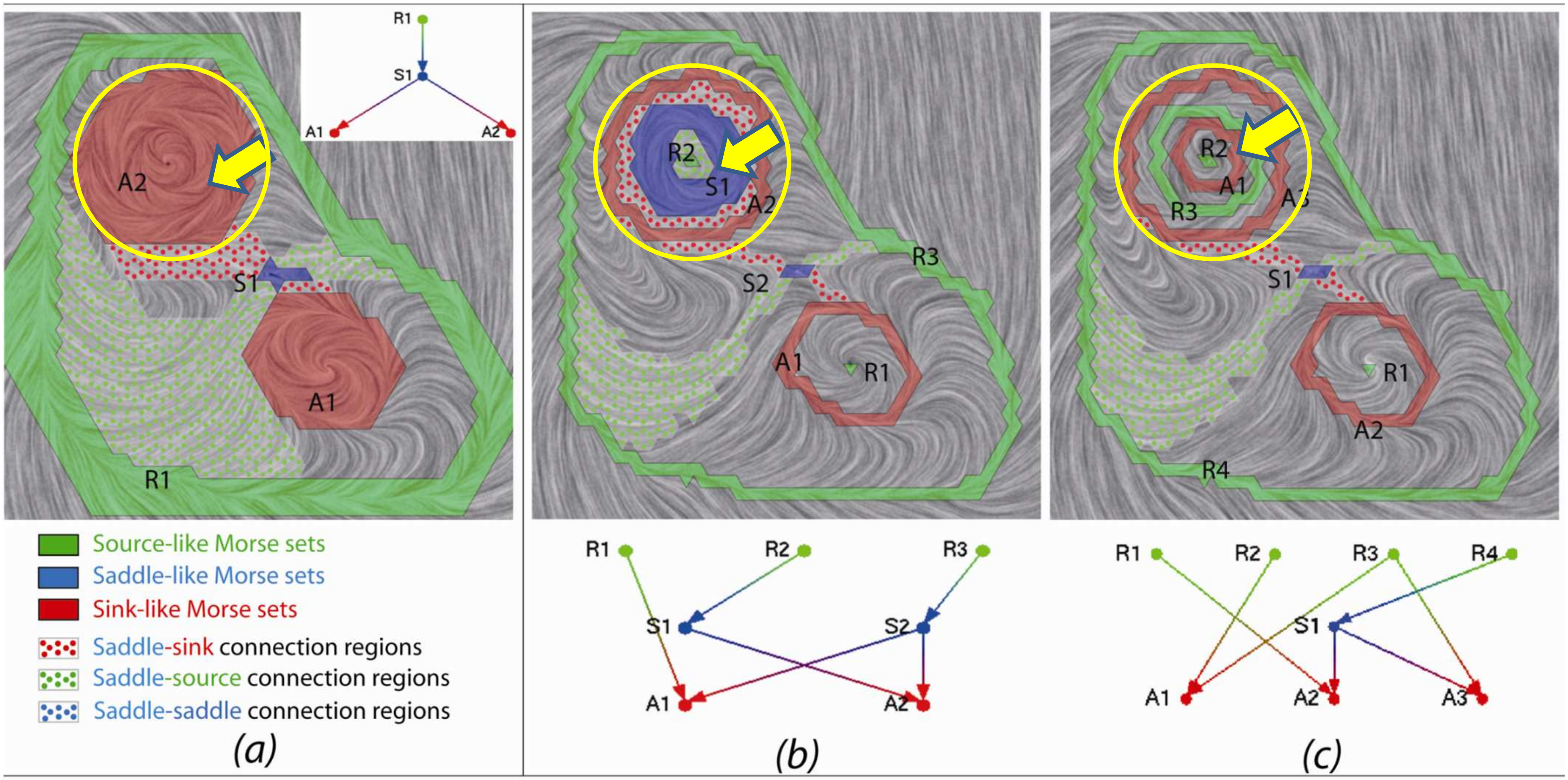
…



τ_n

time_n

Issue (II) With Uniform τ

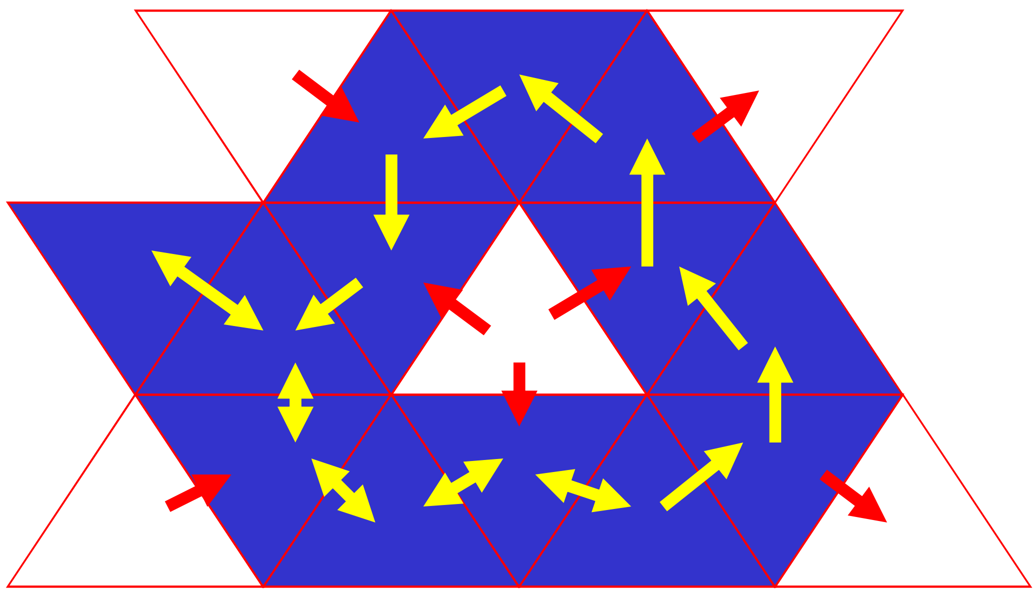


Recall an Important property of Morse decomposition

- The flow recurrent dynamics is only located within Morse sets!

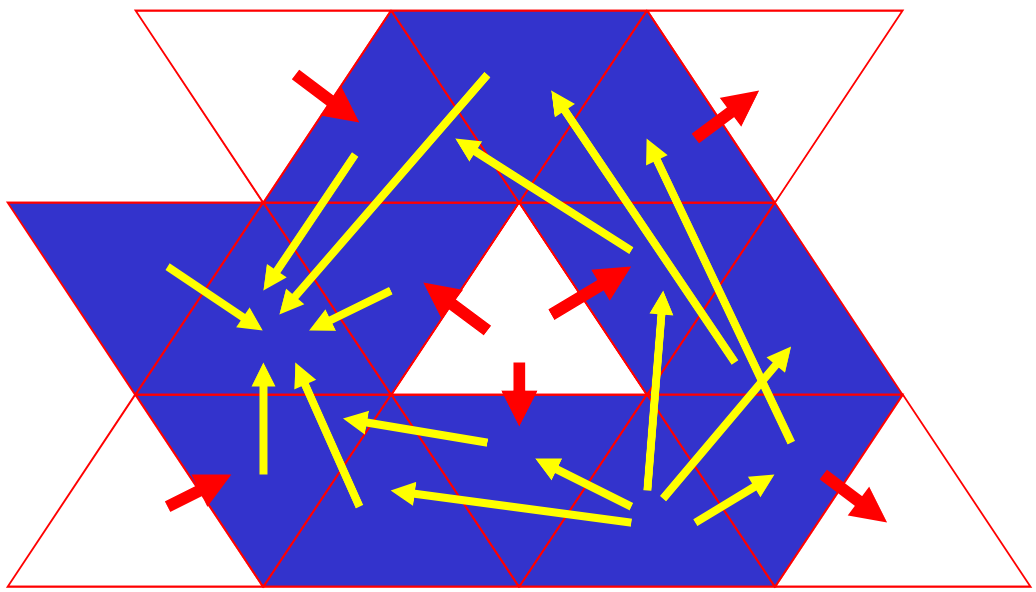
Refinement of A Morse Set

- Remove the inner edges



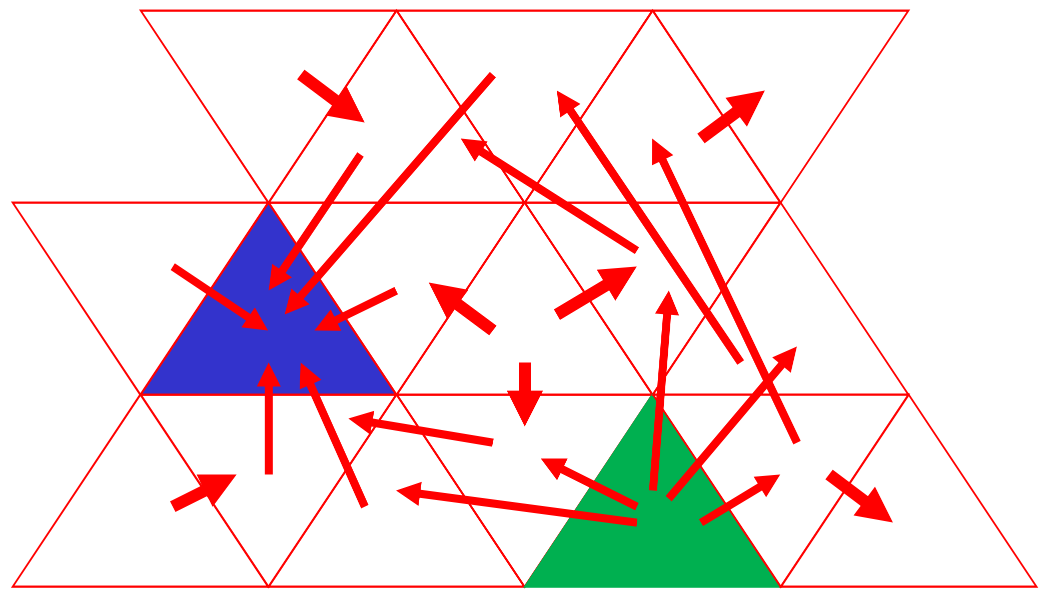
Refinement of A Morse Set

- Replace them with edges computed using a larger τ value

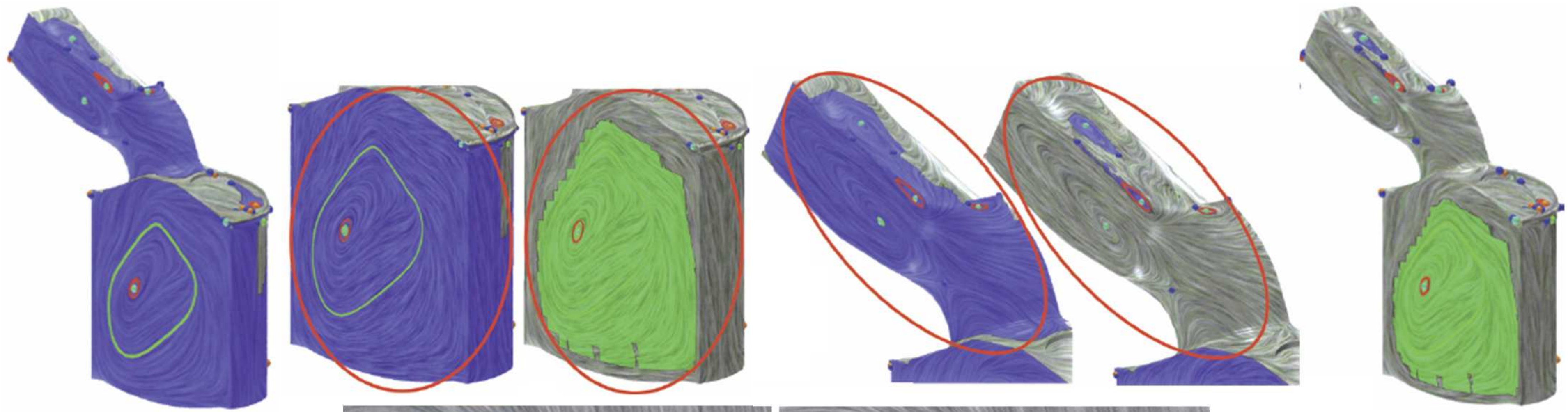


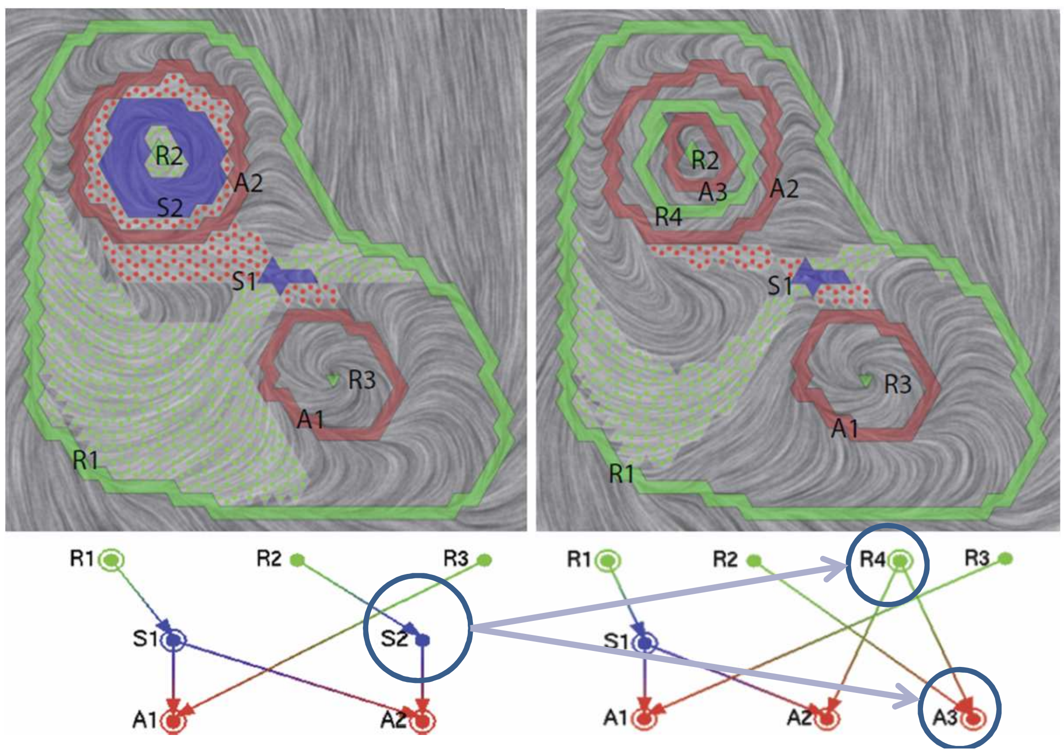
Refinement of A Morse Set

- The refined Morse sets



The Hierarchical Refinement

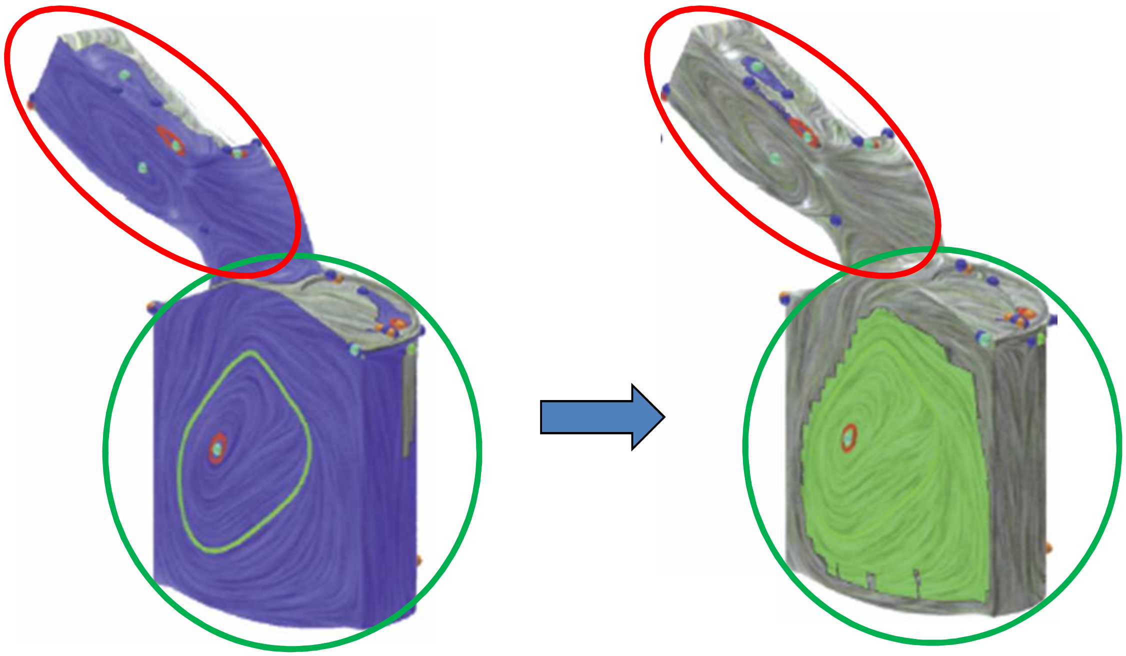




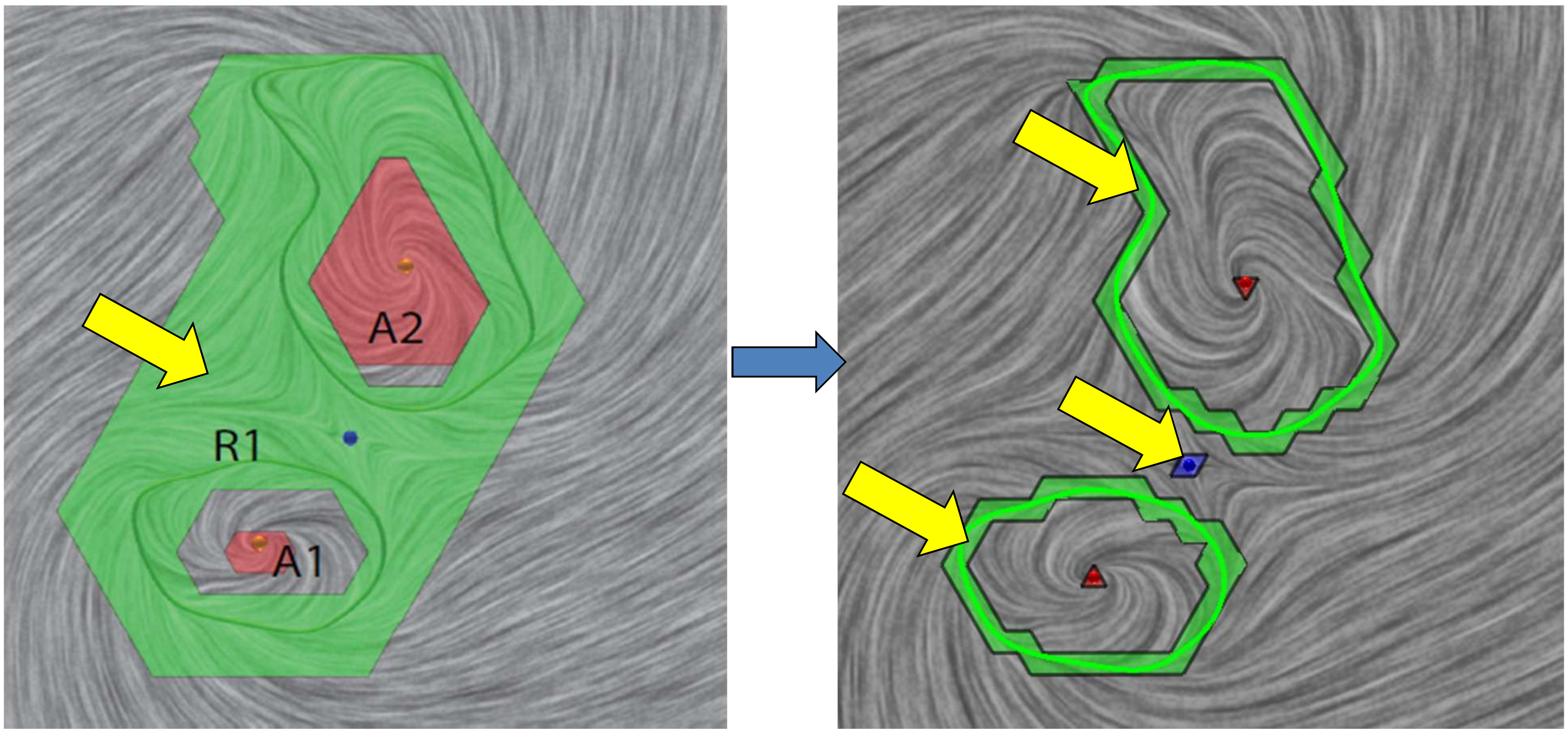
Questions

- Which Morse sets need to be refined?
- How to determine their refinement order?

Geometric Metric



Topology Metric



Priority Values

Given a Morse set M, its priority value for refinement is computed as

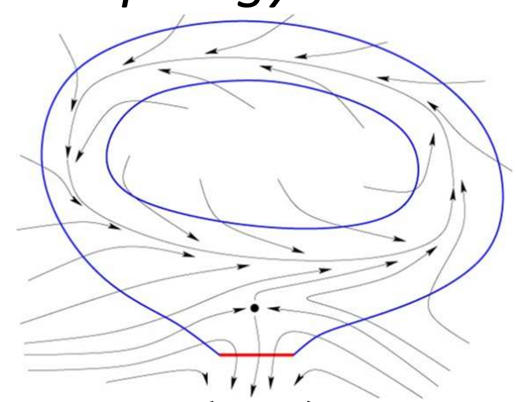
$$P(M) = (1 + tm(M)) \times gm(M)$$

$gm(M)$ is simply the number of triangles in the Morse set M i.e. a *geometry metric*

$tm(M)$ measures the distance between the Conley index of the Morse set M to the basic indices, i.e. a *topology metric*

$$tm(M) = min_{(\beta_0,\beta_1,\beta_2)\in\mathcal{E}} \sum_{k=0}^{2} \left|\beta_k(M) - \beta_k\right|$$

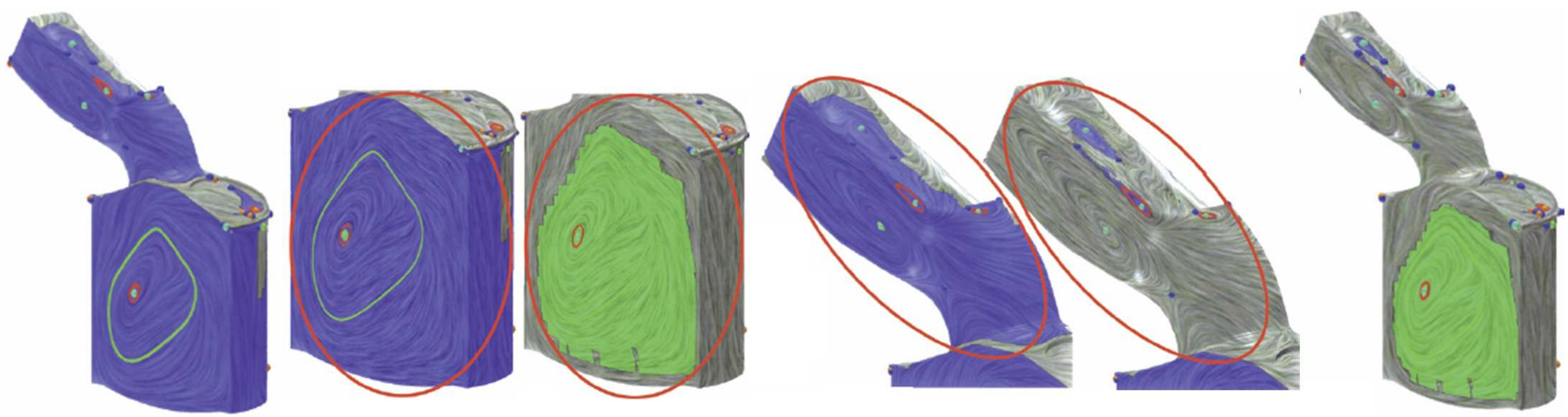
$\mathcal{E} = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1)\}$



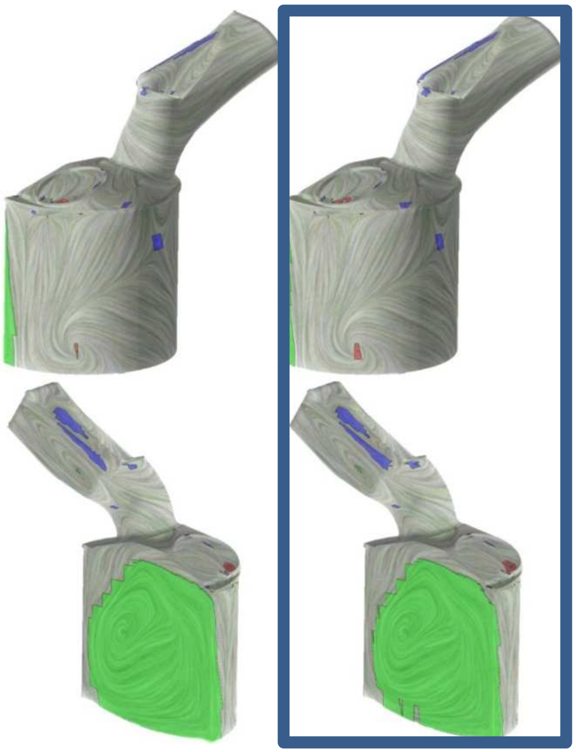
(1,2,0)

The Complete Hierarchical Framework

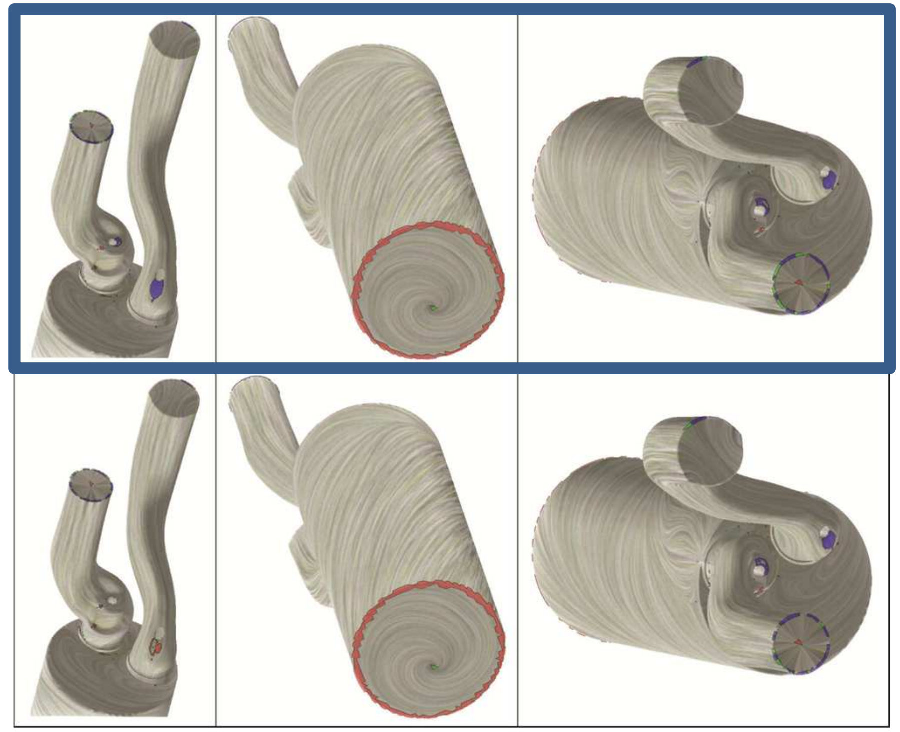
- Compute an MCG using a geometry based method
- Estimate the ***Conley index*** of each Morse set
- Compute a ***priority value*** for each Morse set and add it to a priority queue *Q*
- Iterative process:



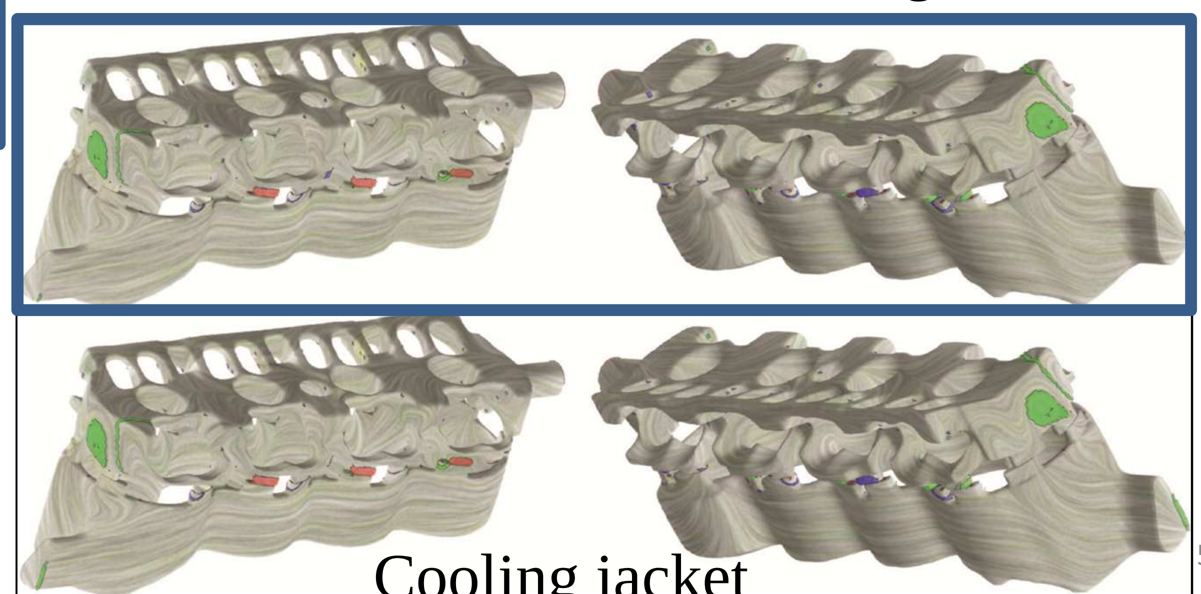
Results



Gas engine



Diesel engine



Cooling jacket

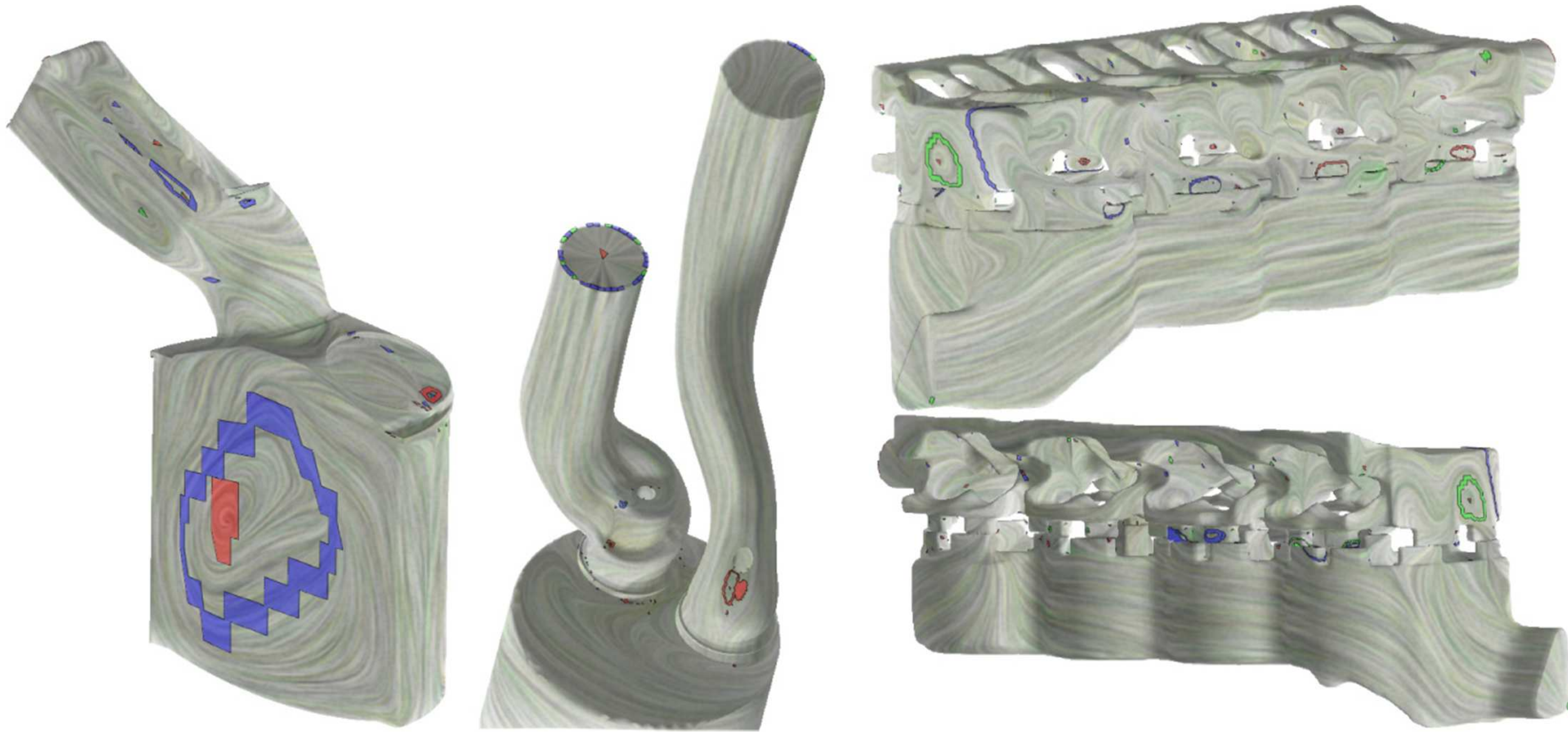
Performance

Dataset	#polygons	Global method τ1,τ2,τ3	time(s)	Hierarchical refinement τ_max	time(s)	*Speed-up*
Gas engine	26,298	0.1, 0.2, 0.4	436.7	0.4	65.97	**6.62**
Diesel engine	221,574	0.1, 0.2, 0.4	1589.3	0.4	96.30	**16.50**
Cooling jacket	227,868	0.1, 0.2, 0.4	2480.2	0.4	435.2	**5.70**

On a PC with Intel(R) Xeon(R) 2.33GHz dual processors and 8GB RAM

- The performance gain at least depends on
  - The complexity of the flow -> number of Morse sets that need refinement
  - The curl of the flow -> the size of each Morse set that needs refinement

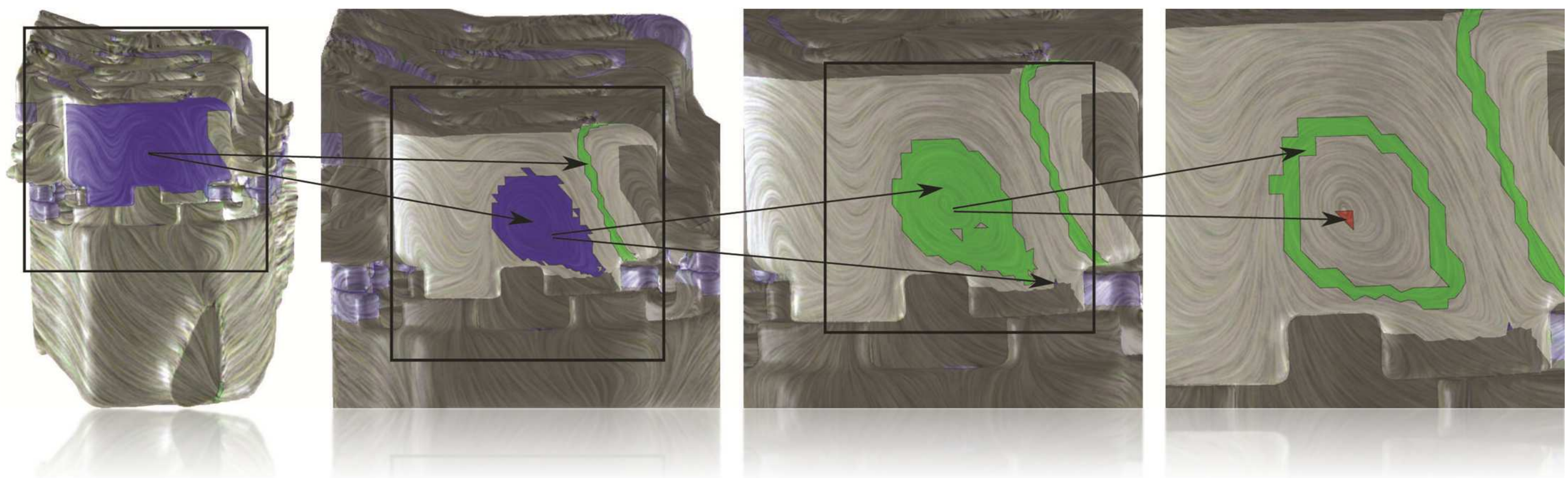
Extension



213s *1012s* *4524s*

The results without setting the maximum τ

Extension



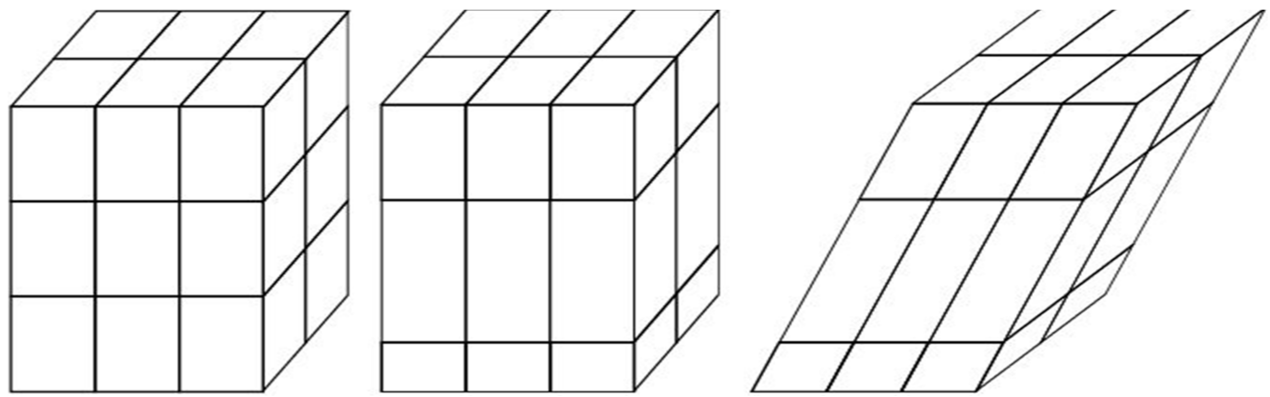
Multi-level representation and visualization

Additional Readings

- Guoning Chen, Konstantin Mischaikow, Robert S. Laramee, and Eugene Zhang. "**Efficient Morse Decompositions of Vector Fields**". IEEE Transactions on Visualization and Computer Graphics, Vol. 14, No. 4, 2008, pp. 848-862.
- Guoning Chen, Qingqing Deng, Andrzej Szymczak, Robert S. Laramee, and Eugene Zhang. "**Morse Set Classification and Hierarchical Refinement using Conley Index**", IEEE Transactions on Visualization and Computer Graphics, Vol 18 (5): pp. 767-782, 2012.

Extend Topology to 3D Steady Vector Field Analysis

Data Structure

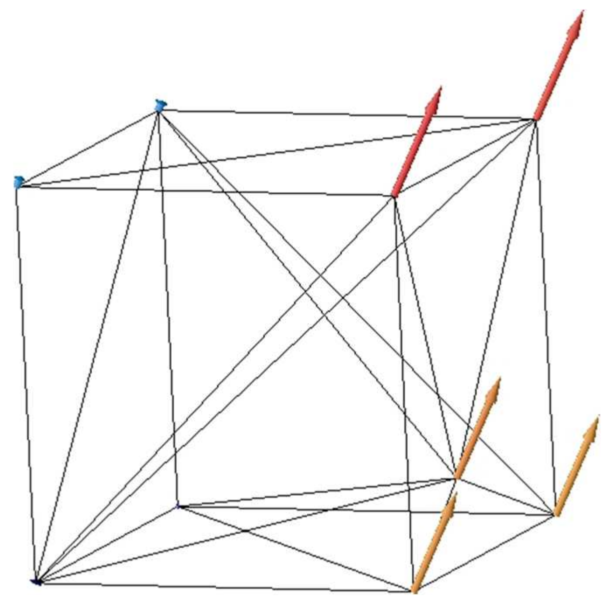


Regular (uniform), rectilinear, and structured grids

Alternative:

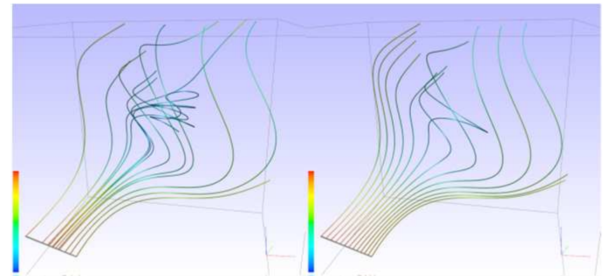
tetrahedral volume elements:

unstructured

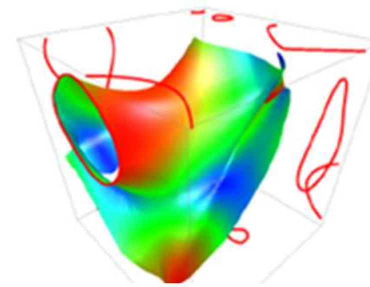


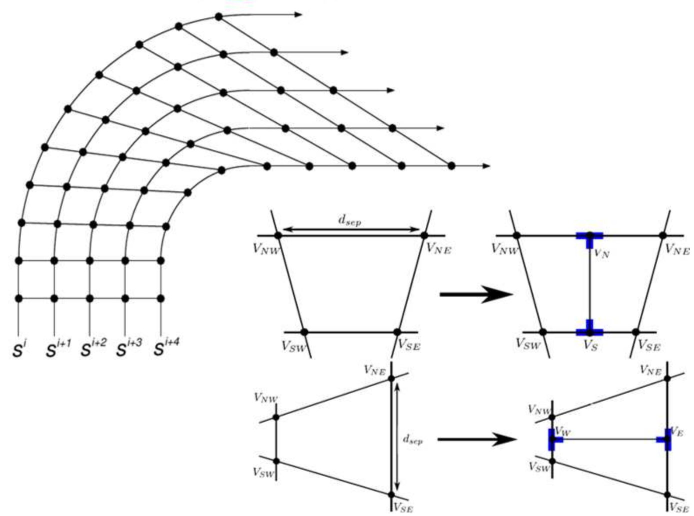
Streamlines and Streamsurfaces

- Streamlines
  - Similar to the 2D definition
  - Computation-wise, more expensive
  - A common objects to visualize 3D flow



- Streamsurfaces
  - A collection of streamlines seeded along a seeding curve in 3D
  - The construction of the surface needs to consider the divergence and convergence of the flow.
  - Typically provide some segmentation of the flow domain as the particles on either side of the streamsurface cannot cross the surface



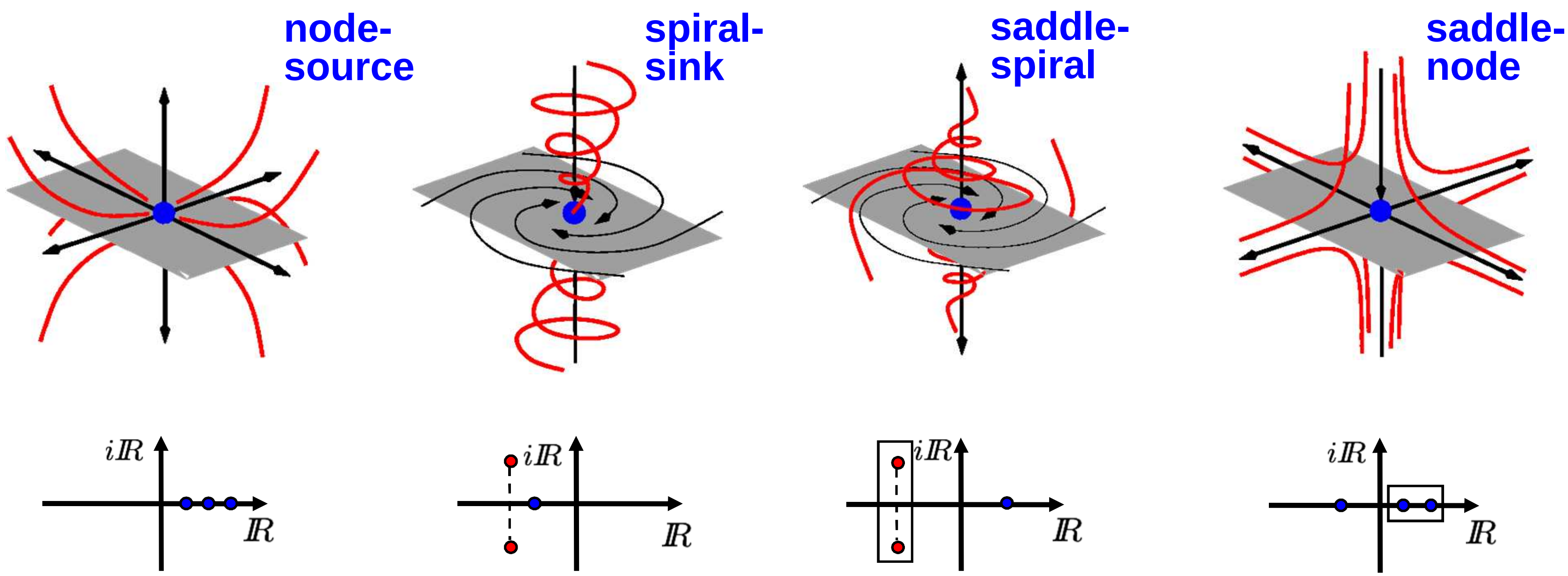


3D Flow Topology

- Similar to 2D case, 3D vector field topology aims to classify the behavior of different streamlines in the domain.
- There are also various flow recurrent dynamics which correspond to those special streamlines, but far more complex than their 2D counterparts
- 3D flow topology again consists of
  - Fixed points
  - Periodic orbits
  - Their connections including separation structures which can now be both streamline and stream surfaces

3D Flow Topology

- Fixed points



- Can be characterized using 3D Poincaré index

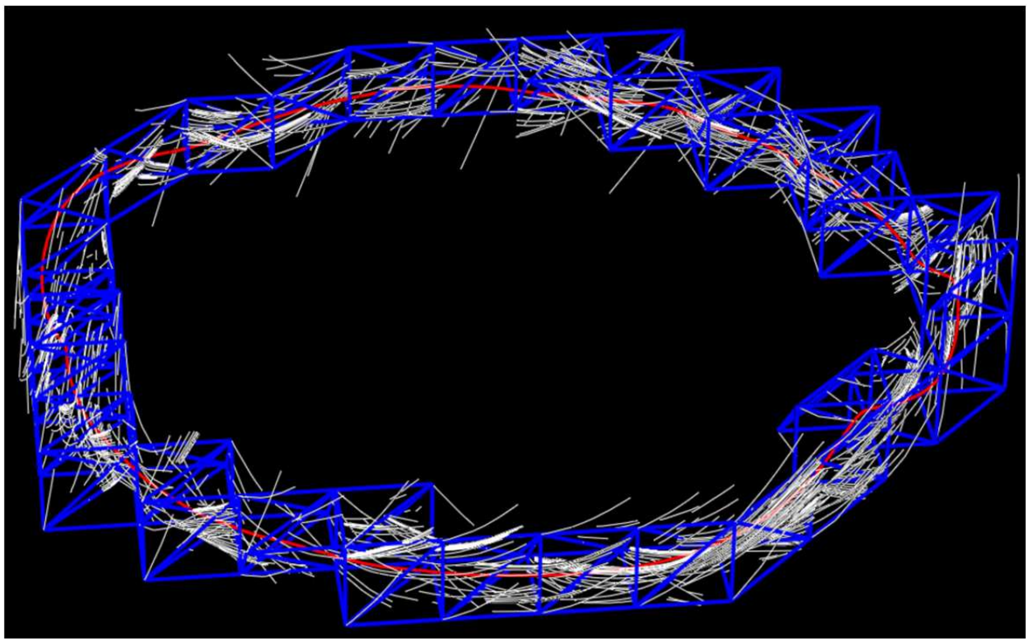
3D Cycles

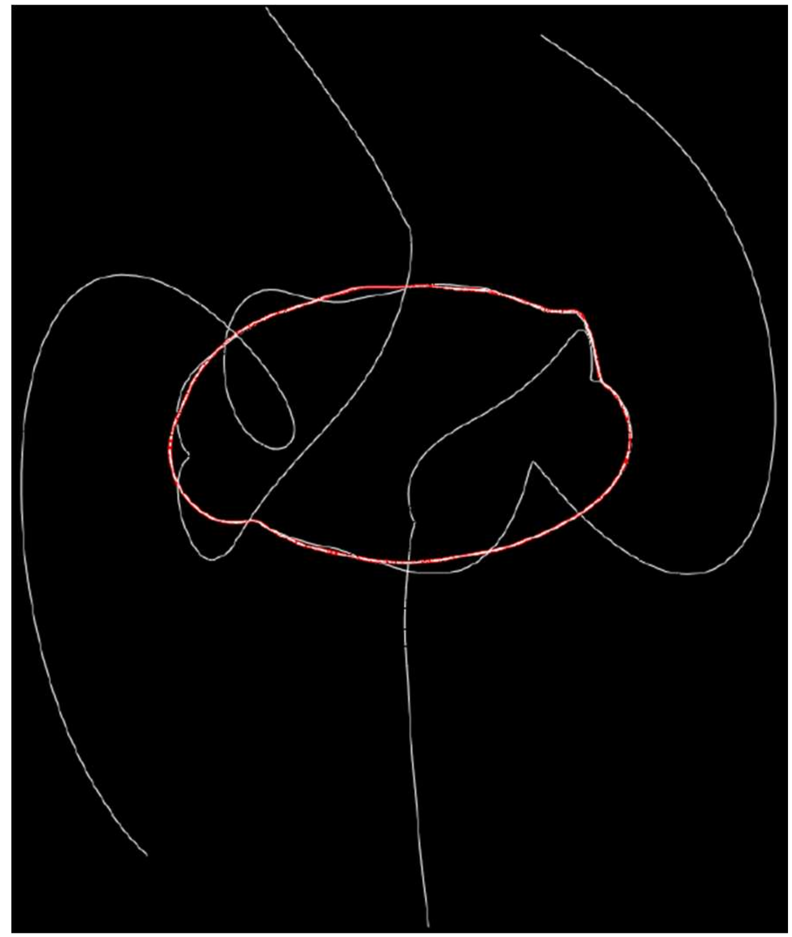
- Similar principle as in 2D
  - Isolate closed cell chain in which streamline integration appears captured
  - Start stream surface integration along boundary of cell-wise region
  - Use flow continuity to exclude reentry cases

Challenging to strange attractor

Source: http://www.stsci.edu/~lbradley/seminar/attractors.html

3D Cycles

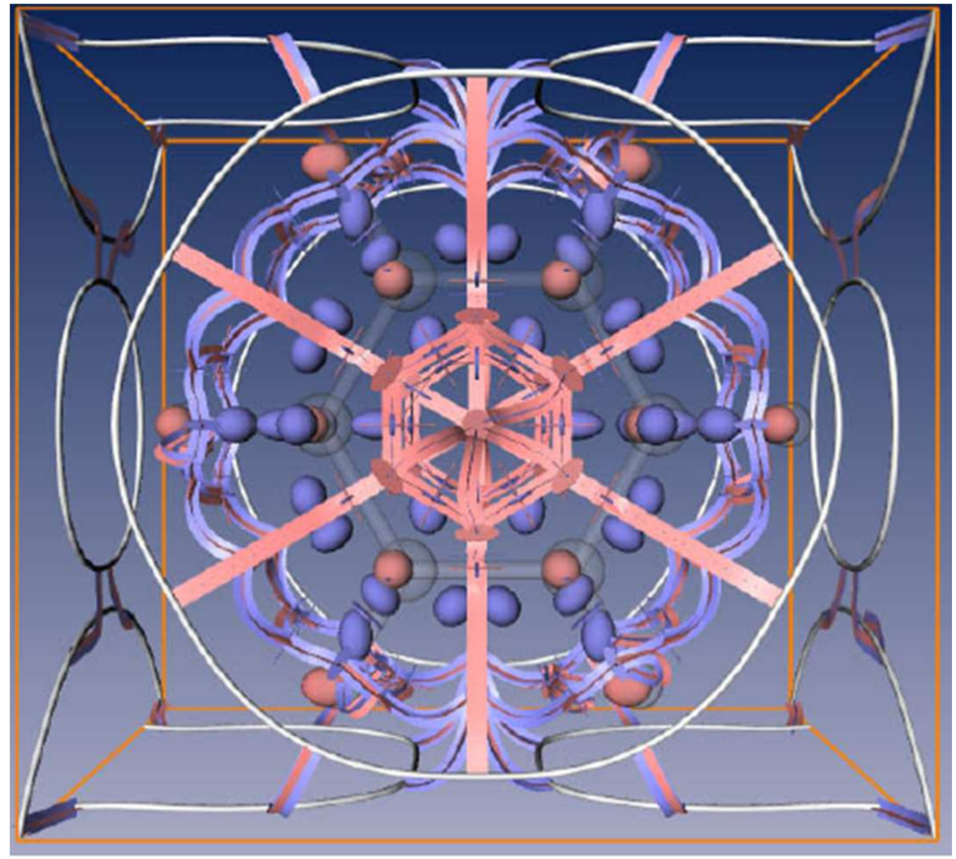




3D Topology Extraction

- Cell-wise fixed point extraction:
  - Compute root of linear / trilinear expression
    - Poincaré index can be applied as well
  - Compute Jacobian at found position
  - If type is saddle compute eigenvectors
- Extract closed streamlines
- Integrate line-type separatrices
- Integrate surface separatrices as stream surfaces
- Find out connection between cycles and fixed points

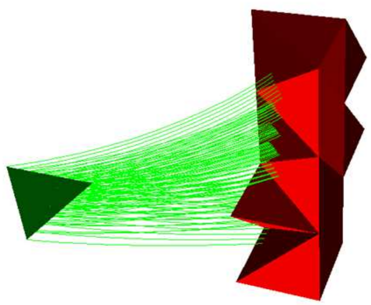
Saddle Connectors

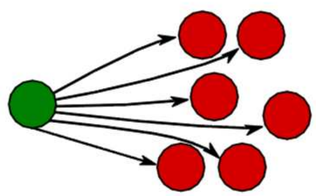


Topological representations of the Benzene data set.
(left) The topological skeleton looks visually cluttered due to the shown separation surfaces.
(right) Visualization of the topological skeleton using saddle connectors.
Source: Weinkauf et al. VisSym 2004

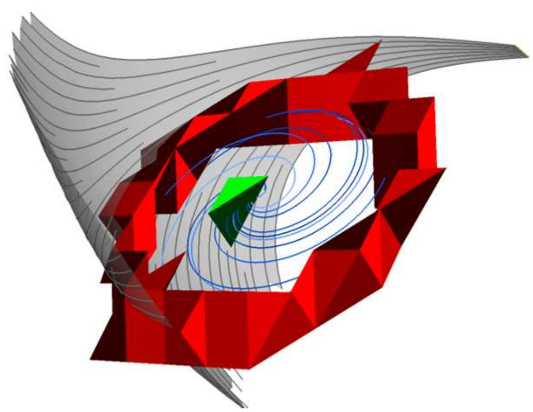
3D Morse Decomposition

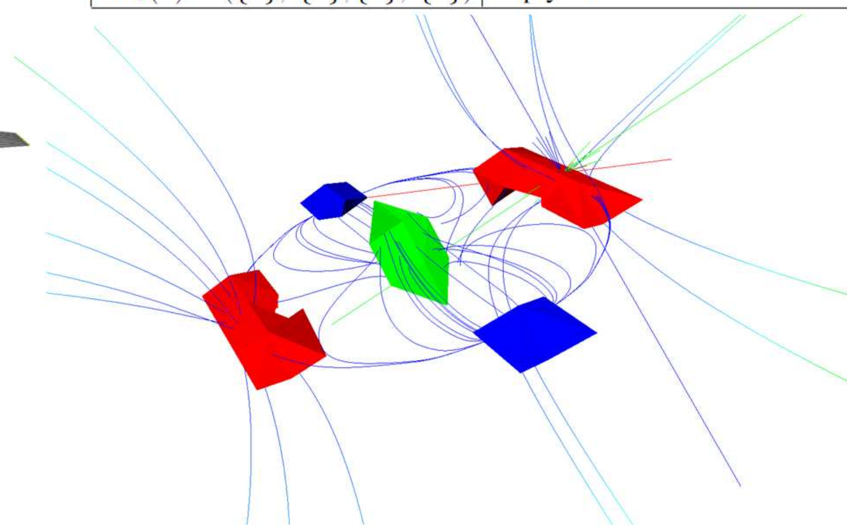
- Similarly, the discrete topology based on Morse decomposition can be directly extended to 3D setting.

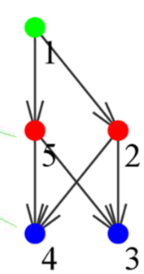




Conley index	flow is equivalent to
$CH_*(x) = (\mathbb{Z}, \{0\}, \{0\}, \{0\})$	attracting fixed point
$CH_*(x) = (\{0\}, \mathbb{Z}, \{0\}, \{0\})$	fixed point with one-dimensional unstable manifold
$CH_*(x) = (\{0\}, \{0\}, \mathbb{Z}, \{0\})$	fixed point with two-dimensional unstable manifold
$CH_*(x) = (\{0\}, \{0\}, \{0\}, \mathbb{Z})$	repelling fixed point
$CH_*(\Gamma) = (\mathbb{Z}, \mathbb{Z}, \{0\}, \{0\})$	attracting closed streamline
$CH_*(\Gamma) = (\{0\}, \mathbb{Z}, \mathbb{Z}, \{0\})$	saddle-like closed streamline
$CH_*(\Gamma) = (\{0\}, \mathbb{Z}_2, \mathbb{Z}_2, \{0\})$	twisted saddle-like closed streamline
$CH_*(\Gamma) = (\{0\}, \{0\}, \mathbb{Z}, \mathbb{Z})$	repelling closed streamline
$CH_*(\emptyset) = (\{0\}, \{0\}, \{0\}, \{0\})$	empty set







Acknowledgment

Thanks for some materials from

- Prof. Robert S. Laramee, Swansea University, UK.
- Dr. Christoph Garth, University of Kaiserslautern, Germany.
- Dr. Holger Theisel, University of Magdeburg, Germany.