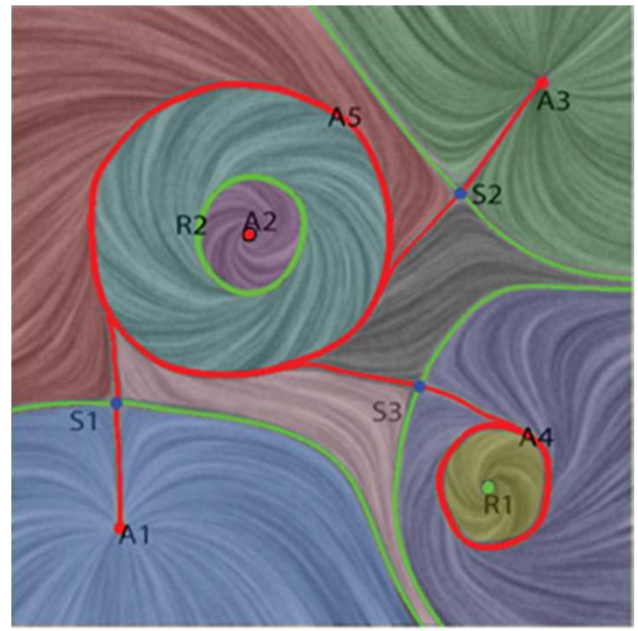
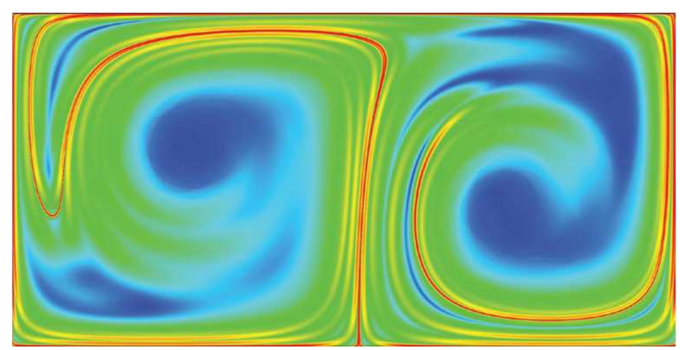
Vector Field Analysis
Other Features

Topological Features

- Flow recurrence and their connectivity



- Separation structure that classifies the particle advection



Vector Field Gradient Recall

- Consider a vector field

$$
{d\mathbf{x}}/{dt} = V(\mathbf{x}) = \vec{\boldsymbol{f}}(x, y, z) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}
$$

- Its gradient is

$$
\nabla V = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{bmatrix}
$$

It is also called the Jacobian matrix of the vector field.
Many feature detection for flow data relies on Jacobian

Divergence and Curl

- **Divergence-** measures the magnitude of outward flux through a small volume around a point

$$div\, V = \nabla \cdot V = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

- **Curl-** describes the infinitesimal rotation around a point

$$curl\, V = \nabla \times V = \begin{bmatrix} \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} & \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} & \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \end{bmatrix}$$

$$\nabla \cdot (\nabla \times V) = 0$$

$$\nabla \times (\nabla \emptyset) = \vec{0}$$

Gauss Theorem

- Also known as divergence theorem, that relates the vectors on the boundary $\partial\mathcal{V} = \mathcal{A}$ of a region $\mathcal{V}$ to the divergence in the region

$$\int_{\mathcal{V}} \; div\,\mathrm{V} d\mathcal{V} = \int_{\mathcal{A}} \mathrm{V} \cdot \mathrm{n} \; d\,\mathcal{A}$$

n being the outward normal of the boundary

- This leads to a physical interpretation of the divergence. Shrinking $\mathcal{V}$ to a point in the theorem yields that the divergence at a point may be treated as the material generated at that point

Stoke Theorem

- The rotation of vector field V on a surface $\mathcal{A}$ is related to its boundary $\partial\mathcal{A} = \mathcal{L}$. It says that the curl on $\mathcal{A}$ equals the integrated field over $\mathcal{L}$.

$$\int_{\mathcal{A}} \mathrm{n} \cdot \mathrm{curl}\,\mathrm{V} d\mathcal{A} = \oint_{\mathcal{L}} \mathrm{V} \cdot dr$$

- This theorem is limited to two dimensional vector fields.

Another Useful Theorem about Curl

- In the book of Borisenko [BT79]
- Suppose $\mathrm{V} = \mathrm{V}' \times \mathbf{c}$ with $\mathbf{c}$, an arbitrary but fixed vector, substituted into the divergence theorem. Using $div(\mathrm{V}' \times \mathbf{c}) = \mathbf{c} \cdot curl\ \mathrm{V}'$, one gets

$$\int_{\mathcal{V}} \mathrm{curl}\ \mathrm{V}' d\mathcal{V} = \int_{\mathcal{A}} \mathrm{n} \times \mathrm{V}' d\mathcal{A}$$

- Stoke's theorem says that the flow around a region determines the curl.

- The second theorem says: Shrinking the volume $\mathcal{V}$ to a point, the curl vector indicates the axis and magnitude of the rotation of that point.

2D Vector Field Recall

- Assume a 2D vector field

$$d\mathbf{x}/_{dt} = V(\mathbf{x}) = \vec{\boldsymbol{f}}(x, y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} ax + by + c \\ dx + ey + f \end{pmatrix}$$

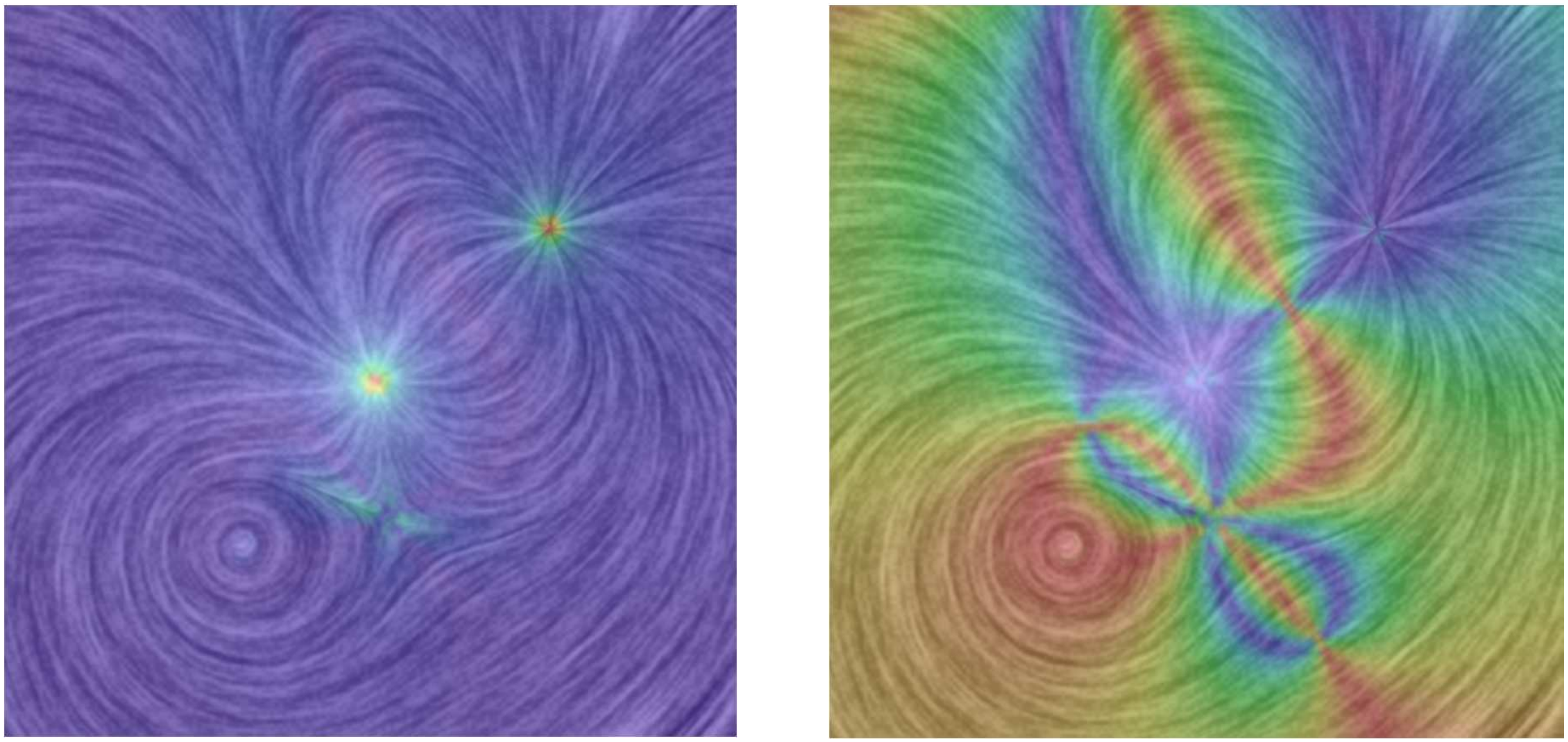
- Its Jacobian is

$$\nabla V = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

- Divergence is $a + e$
- Curl is $\mathrm{b} - \mathrm{d}$

Given a vector field defined on a discrete mesh, it is important to compute the coefficients a, b, c, d, e, f for later analysis.

Examples of Divergence and Curve of 2D Vector Fields



Divergence and curl of a vector field

Potential or Irrotational Fields

- A vector field V is said to be a potential field if there exists a scalar field φ with
$$\mathrm{V} = \mathrm{grad}\,\varphi = \nabla\varphi$$

φ is called the scalar potential of the vector field V

- A vector field V living on a simply connected region is irrotational, i.e. $\mathrm{curl}\;\mathrm{V} = 0$ (i.e. curl-free), if and only if it is a potential field.

- It is worth noting that the potential defining the potential field is not unique, because
$$\mathrm{grad}(U + c) = \mathrm{grad}\,U + \mathrm{grad}\,c = \mathrm{grad}\,U + 0 = \mathrm{grad}\,U$$

Solenoidal Fields

- Or divergence-free field

$$\mathrm{V} = \mathrm{curl}\,\Phi = \nabla \times \Phi$$

- Solenoidal fields stem from potentials too, but this time from **vector potentials**, Φ.

- These fields can describe incompressible fluid flow and are therefore as important as potential fields.

- A vector field V is solenoidal, i.e. $V = \nabla \times \Phi$ with $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$, if and only if the divergence of V vanishes.

- The vector potential here is not unique as well

$$\mathrm{curl}\,(\mathrm{V} + \nabla U) = \mathrm{curl}\,\mathrm{V} + \mathrm{curl}\,\nabla U = \mathrm{curl}\,\mathrm{V} + 0 = \mathrm{curl}\,\mathrm{V}$$

Laplacian Fields

- A vector field V which is both potential and solenoidal (i.e. both curl-free and divergence-free), is called a Laplacian field.

- In a simply connected region, a Laplacian field is the gradient of a scalar potential which satisfies Laplace differential equation $\Delta\varphi = 0$.

- Scalar function like φ whose Laplacian vanishes, are called **harmonic functions**.
  - They are completely determined by their boundary values.
  - There exists one function satisfying Laplace's equation for fixed boundary values.

Helmholtz Decomposition

$$V = \nabla\varphi + \nabla \times \Phi$$

Curl (or rotation) free

Divergence free

Hodge decomposition

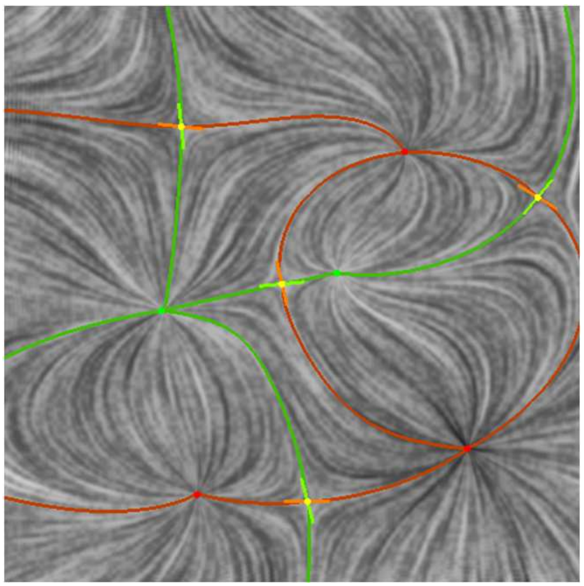
$$V = \nabla\varphi + \nabla \times \Phi + \gamma$$

Curl (or rotation) free

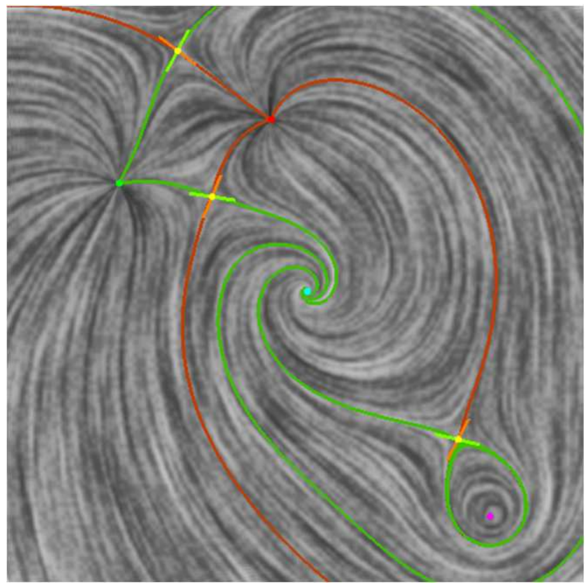
Divergence free

Harmonic

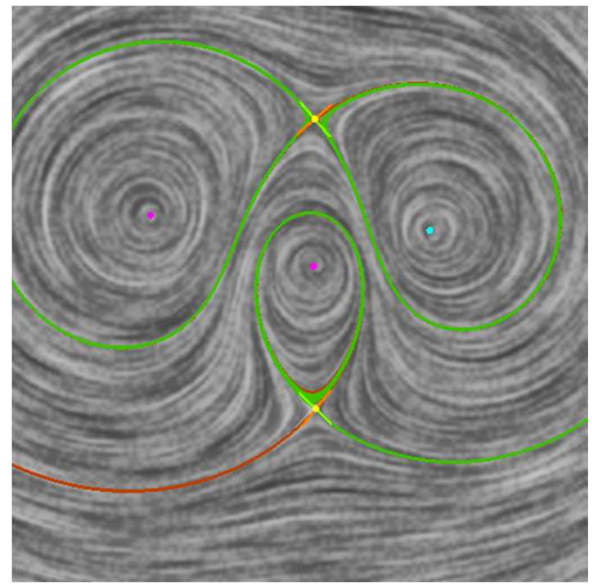
Helmholtz Decomposition Example



curl-free



neither



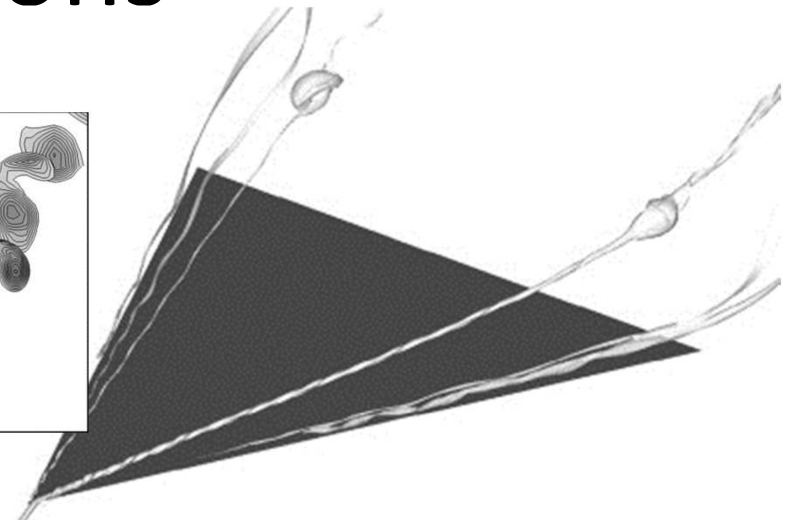
divergence-free

General Feature Classifications

- Points
  - Fixed points, vortex centers
- Lines
  - Features that occupy a set of points forming a line
  - 3D vortex cores, ridge lines, separation/attachment lines, cycles
- Surfaces
  - Features cover a set of points representing a surface
  - Shock wave, iso-surfaces, separation surfaces in 3D
- Volume
  - Features cover a non-zero region in 3D
  - Vortex region, 3D Morse sets, coherent structure

One important non-topological features in vector fields is vortex

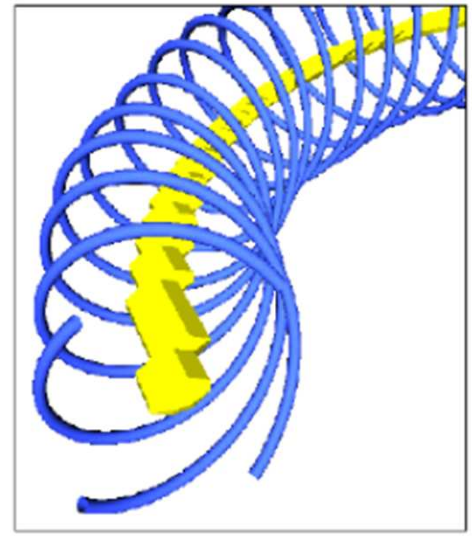
Applications



Vortex Definition

- No rigorous and widely-accepted definition
- Capturing some swirling behavior

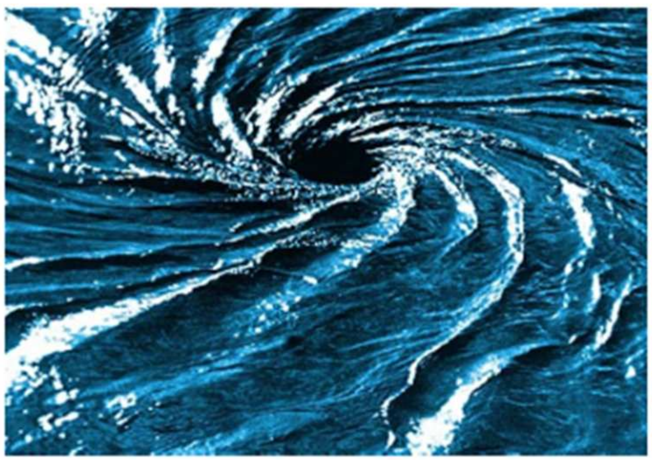
- Robinson 1991:
  - “A vortex exists when instantaneous **streamlines** mapped onto a plane normal to the vortex core exhibit a roughly circular or spiral pattern, when viewed from a reference frame moving with the center of the vortex core”

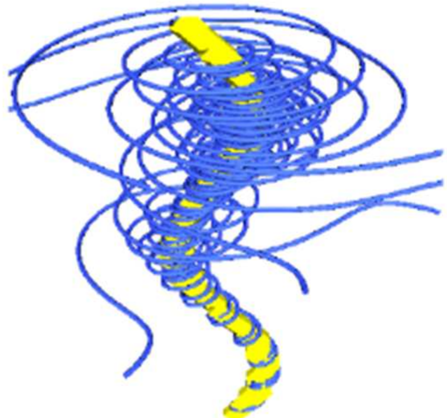


- Requires a priori detection
- Not always Galilean invariant: varying by adding constant vector fields

Different Definitions

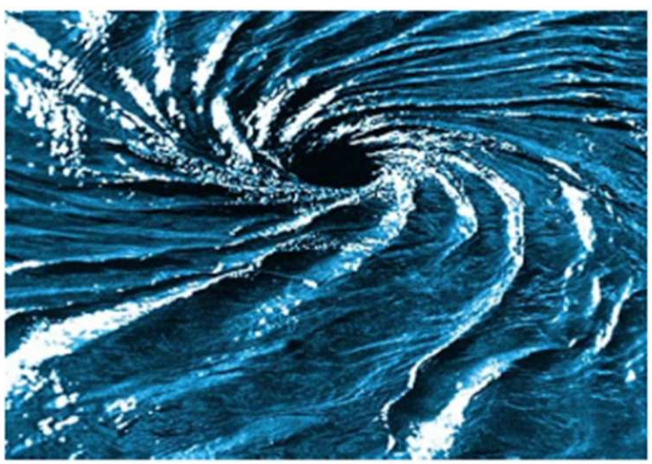
- A vortex?
- [lugt'72]
  - A vortex is the rotating motion of a multitude of material particles around a common center
  - Vorticity is sufficiently strong – not enough to detect

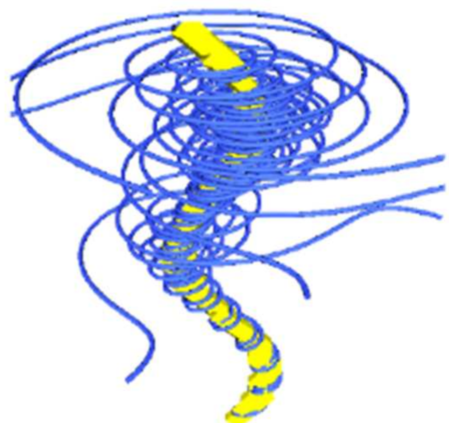




Different Definitions

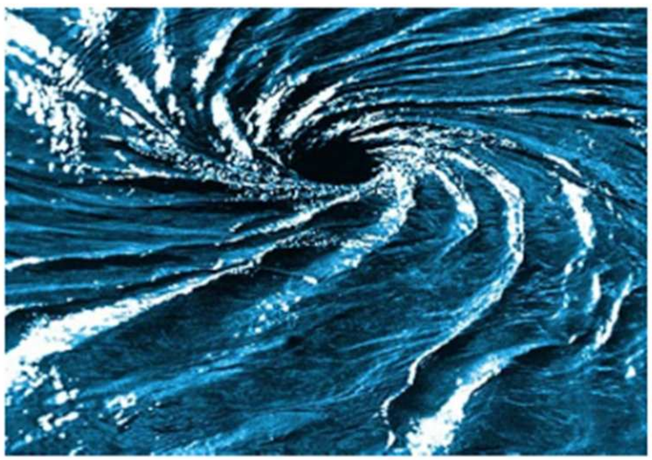
- A vortex?
- [Robinson’91]
  - A vortex exists when its streamlines, mapped onto a plane normal to its core, exhibit a circular or spiral pattern, under an appropriate reference frame

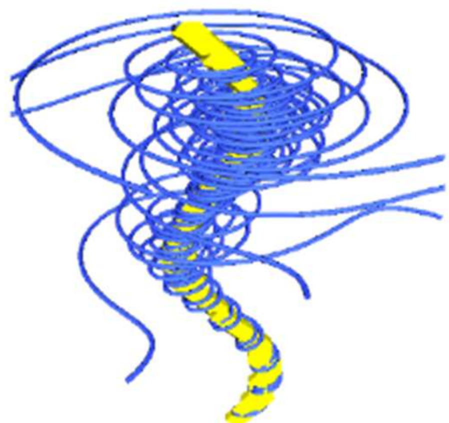




Different Definitions

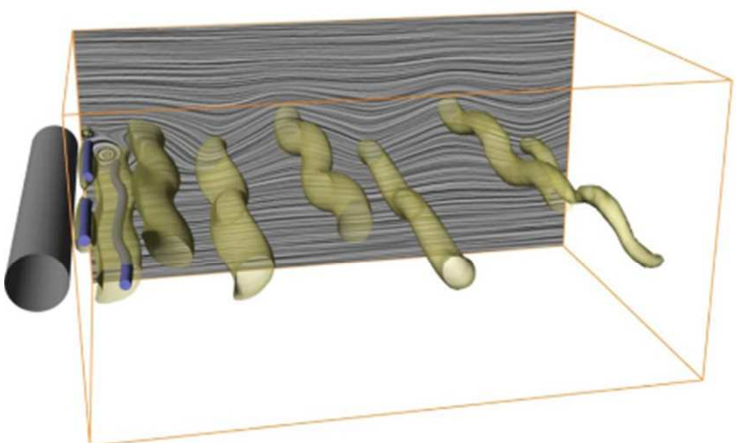
- A vortex?
- [Portela’97]
  - A vortex is comprised of a central core region surrounded by swirling streamlines



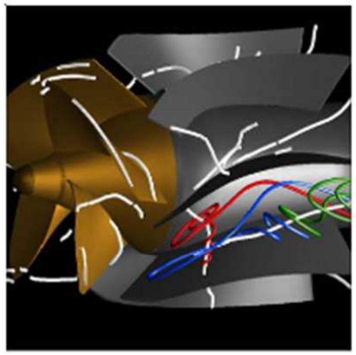


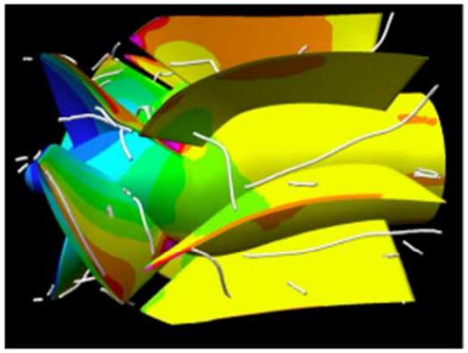
Vortex Structures

- Two main classes of vortex structures
  - Region based methods: isosurfaces of scalar fields
  - Line based methods: extract vortex core lines



region based





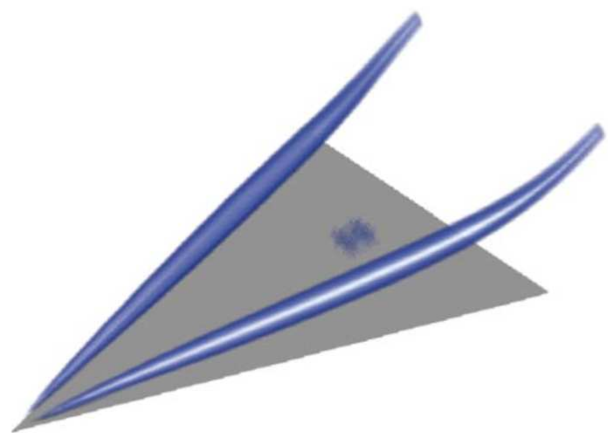
line based

Region Based

- Threshold on **pressure**:

$$p \leq p_{thresh}$$

- Idea: centripetal force induces pressure gradient
  - Very easy to implement and compute
  - Purely local criterion
- Problems:
  - Arbitrary threshold
  - Pressure can vary greatly along a vortex

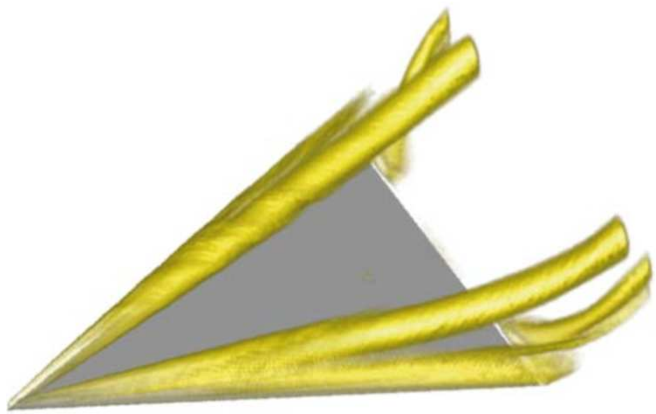


Region Based

- Threshold on **vorticity magnitude**:

$$|\nabla \times V| \geq \omega_{thresh}$$

- Idea: strong infinitessimal rotation
  - Common in fluid dynamics community
  - Very easy to implement and compute, purely local
- Problems:
  - Arbitrary threshold
  - Vorticity often highest near boundaries
  - Vortices can have vanishing vorticity

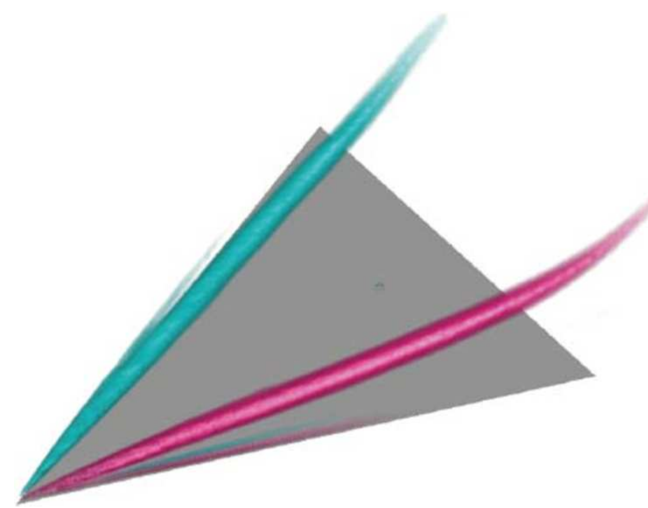


Region Based

- Threshold on (normalized) helicity magnitude

$$(\nabla \times V) \cdot V \geq h_{thresh}$$

- Idea: use vorticity but exclude shear flow
  - Still easy to implement and compute, purely local
- Problems:
  - Arbitrary threshold
  - Fails for curved shear layers
  - Vortices can have vanishing vorticity



Region Based

- λ_2-criterion

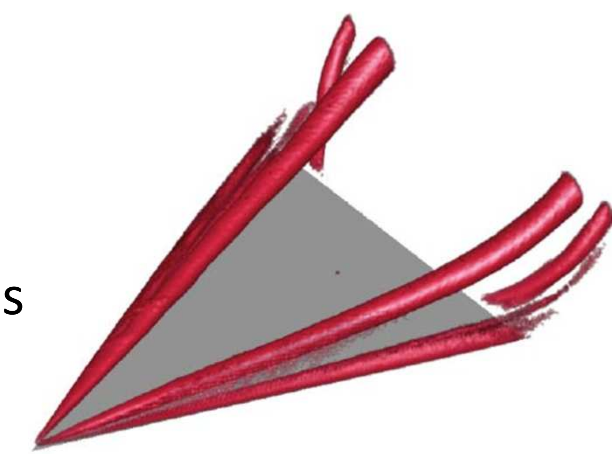
$$S := \frac{1}{2}(\mathrm{J} + J^T)$$

Shear contribution of J

$$\Omega := \frac{1}{2}(\mathrm{J} - J^T)$$

rotational contribution of J

  - Define as the largest eigenvalues of $S^2 + \Omega^2$
  - Vortical motion where $\lambda_2 < 0$
    - Precise threshold, nearly automatic
    - Very widely used in CFD
    - Susceptible to high shear
    - Insufficient separation of close vortices



Region Based

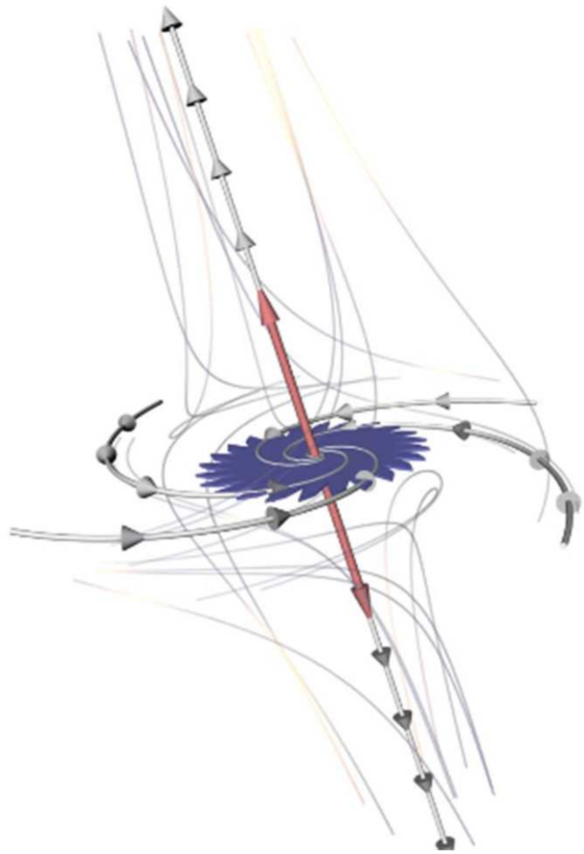
- Q-criterion (Jeong, Hussain 1995)
- Positive **2nd invariant** of Jacobian

$$Q = \frac{1}{2}(\|\Omega\|^2 - \|S\|^2)$$

- Idea: $Q > 0$ implies local pressure smaller than surrounding pressure. Condition can be derived from characteristic polynomial of the Jacobian.
  - Common in CFD community
  - Can be physically derived from kinematic vorticity (Obrist, 1995)
  - Need good quality derivatives, can be hard to compute

Line Based

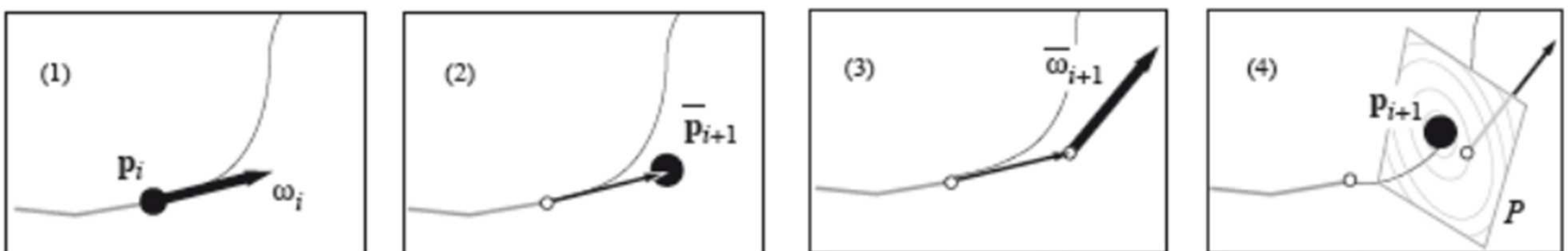
- Separation lines starting from focus saddle critical points [Globus/Levit 92]



Line Based

- Banks-Singer (1994):

- Idea: Assume a point on a vortex core is known.
  - Then, take a step in vorticity direction (predictor).
  - Project the new location to the pressure minimum perpendicular to the vorticity (corrector).
  - Break if correction is too far from prediction



Compute the vorticity at a point on the vortex core. | Step in the vorticity direction to predict the next point. | Compute the vorticity at the predicted point. | Correct to the pressure min in the perpendicular plane.

Image from Banks, Singer, Vis 1994

Line Based

- Banks-Singer, continued

- Results in core lines that are roughly vorticity lines and pressure valleys.

  – Algorithmically tricky
  – Seeding point set can be large (e.g. local pressure minima)
  – Requires additional logic to identify unique lines

Line Based

- [Sujudi, Haimes 95]

- In 3D, in areas of 2 imaginary eigenvalues of the Jacobian matrix: *the only real eigenvalue is parallel to* V

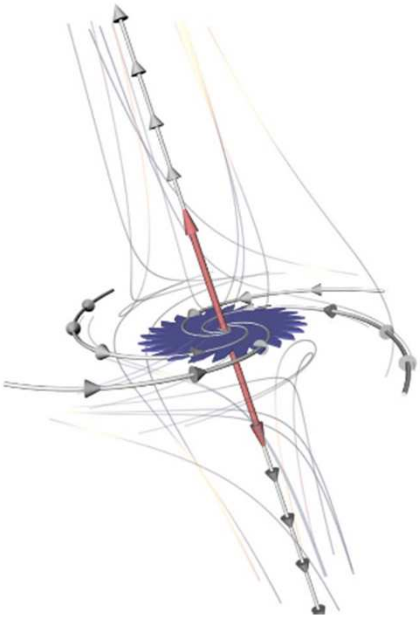
  - In practice, standard method in CFD, has proven successful in a number of applications
  - Criterion is local per cell and readily parallelized
  - Resulting line segments are disconnected (Jacobian is assumed piecewise linear)
  - Numerical derivative computation can cause noisy results
  - Has problems with curved vortex core lines (sought-for pattern is straight)

Line Based

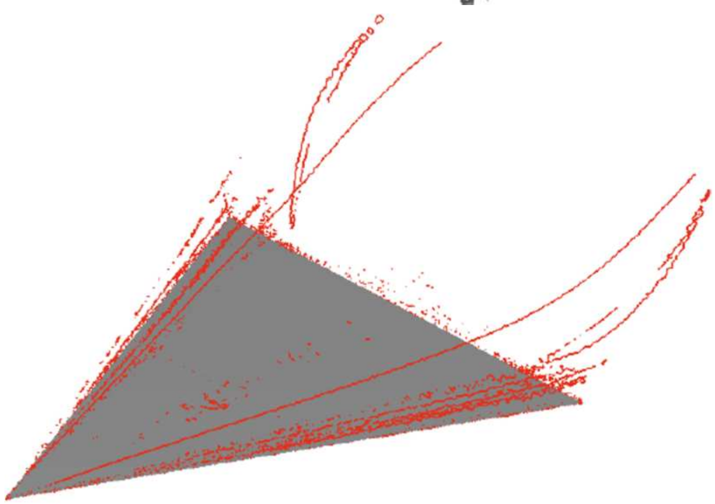
- Eigenvector method [Sujudi and Haimes 95]

$$w(\mathbf{x}) = v(\mathbf{x}) - \big(v(\mathbf{x}) \cdot e(\mathbf{x})\big)e(\mathbf{x})$$
$$w(\mathbf{x}) = 0$$

Reduced velocity



Although a point $\mathbf{x}$ on the core structure is surrounded by spiraling integral curves, the flow vector at $\mathbf{x}$ itself is solely governed by the non-swirling part of the flow.

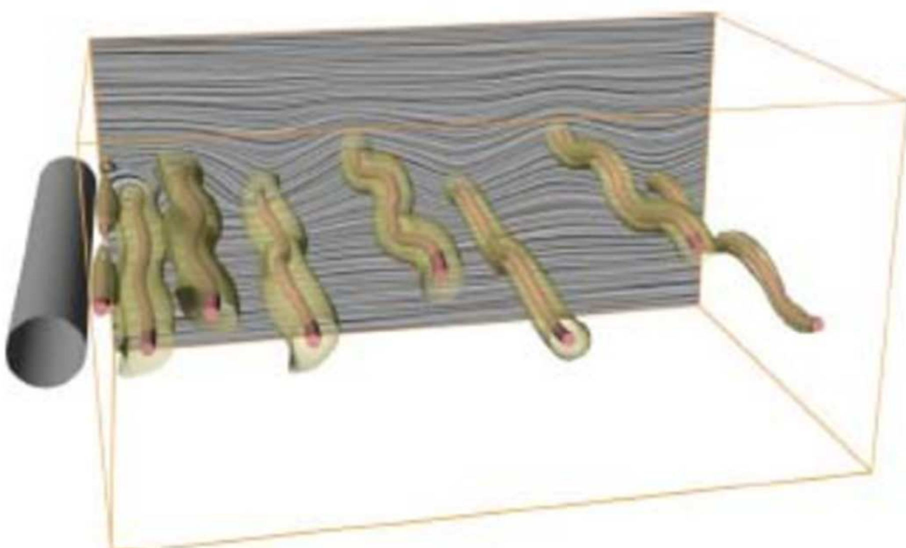


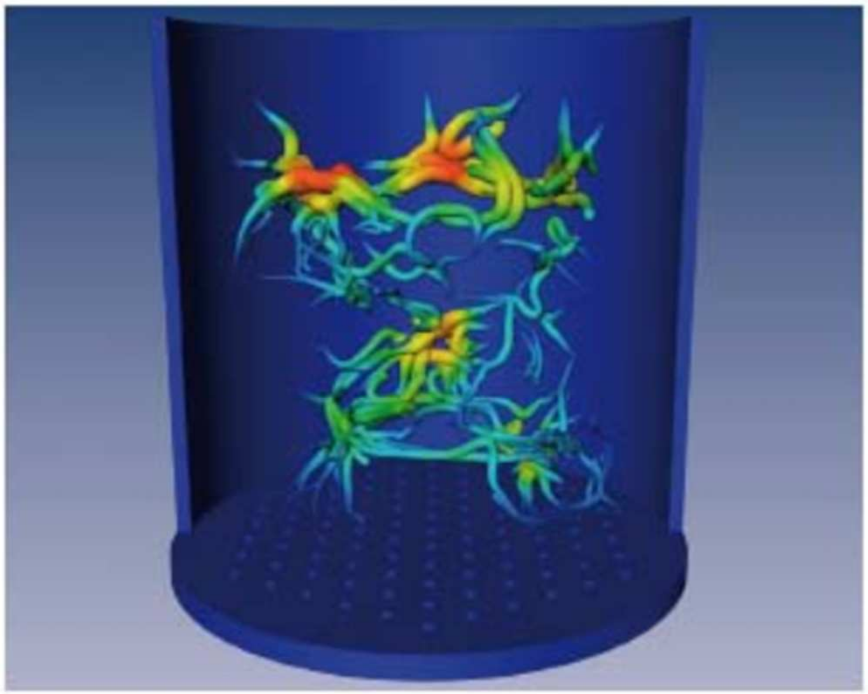
Line Based

- Sahner et al. 2005
- Idea: construct a special vector field that allows to model ridge/valley-lines as integral curves ("feature flow field").
- Authors applied it to Q-criterion and λ_2–criterion.
  - Works well in practice
  - Feature flow field requires high-order partial derivatives that are difficult to compute in certain data sets
  - Seed point set required (usually minimal points)

Line Based

- Sahner et al., results





Images from Sahner et al., Eurovis 2005

Line Based

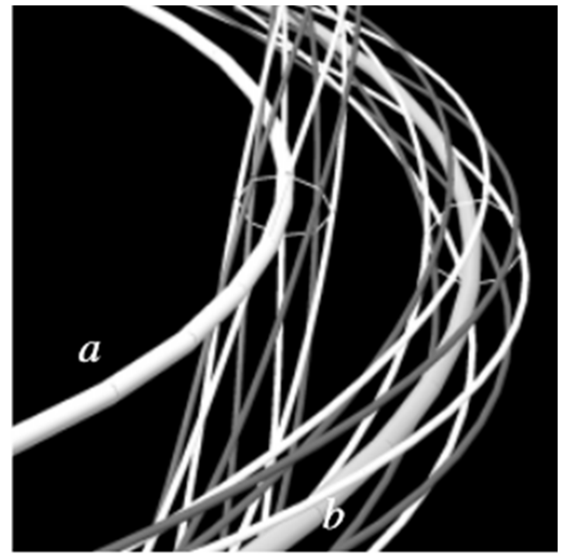
- The parallel vector operator
  [Roth and Peikert98]

- Given: 3D vector field V
- The curvature vector of $\boldsymbol{v}$ is

$$\boldsymbol{c} = \frac{\boldsymbol{v} \times \boldsymbol{a}}{|\boldsymbol{v}|^3}$$

  where $\boldsymbol{a} = \frac{D\boldsymbol{v}}{Dt}$ is the acceleration of

  Let $\boldsymbol{b} = \frac{D^2\boldsymbol{v}}{Dt^2}$



- Vortex core line: all locations in the domain where $\boldsymbol{b}$ is parallel to $\boldsymbol{v}$,

- Line structures

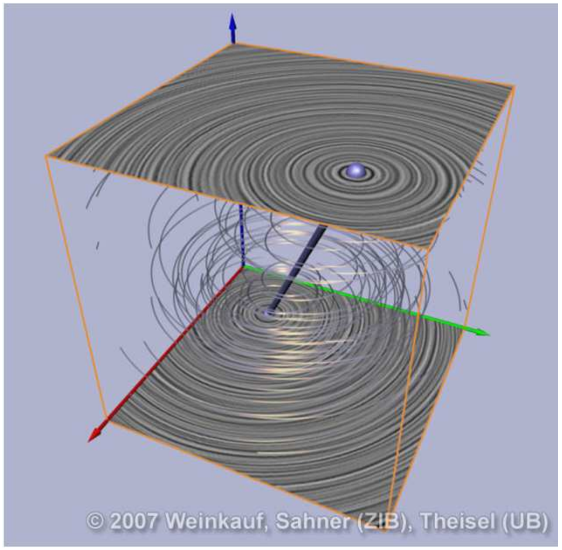
- Particular parallel vector approaches

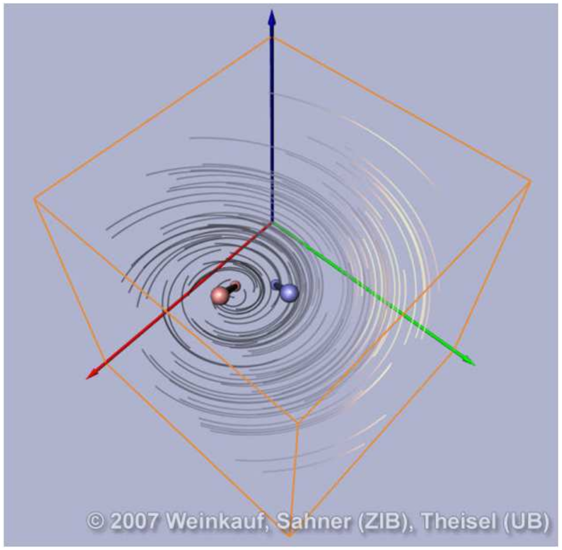
Authors	*Type *)*	*Description*	*Basic Formula*	*Additional Criteria*	*Page*
Sujudi, Haimes	Z	zero curvature lines of velocity	$\boldsymbol{v} \parallel (\nabla \boldsymbol{v})\, \boldsymbol{v}$	complex eigenvalues	80
Miura, Kida	Z,E	valley lines of pressure	$\boldsymbol{v} \parallel (\nabla \boldsymbol{v})\, \boldsymbol{v}$ with $\boldsymbol{v} = \nabla p$	valley line (conditions on eigenvalues)	87
Strawn, Kenwright, Ahmad	E	maximum lines of vorticity	$\boldsymbol{w} \parallel (\nabla \boldsymbol{w})^T \boldsymbol{w}$ with $\boldsymbol{w} = \nabla \times \boldsymbol{v}$	maximum line (conditions on eigenvalues)	79
Levy, Degani, Seginer		velocity parallel to vorticity	$\boldsymbol{v} \parallel \nabla \times \boldsymbol{v}$	rotation (non-zero vorticity)	74
Banks, Singer		vorticity parallel to gradi-ent of pressure	$\nabla p \parallel \nabla \times \boldsymbol{v}$	minimum of pressure (condition on eigenvalues)	77

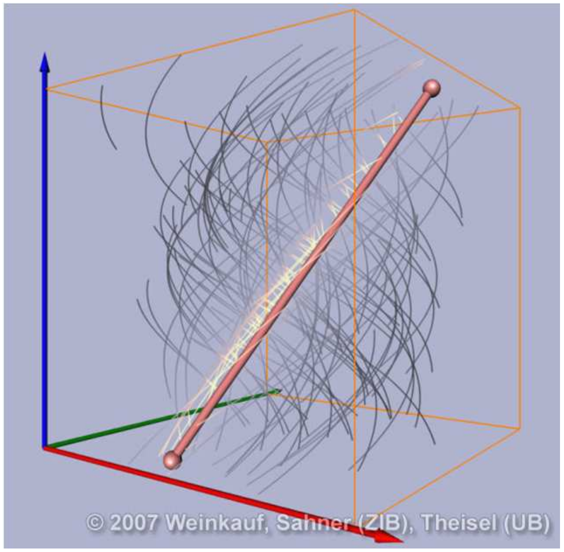
From M. Roth’s PhD thesis

Extend to Unsteady Flow

- Path lines: cores of swirling particle motion
  [Weinkauf et al., Vis 2007]







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