Sub-Topics

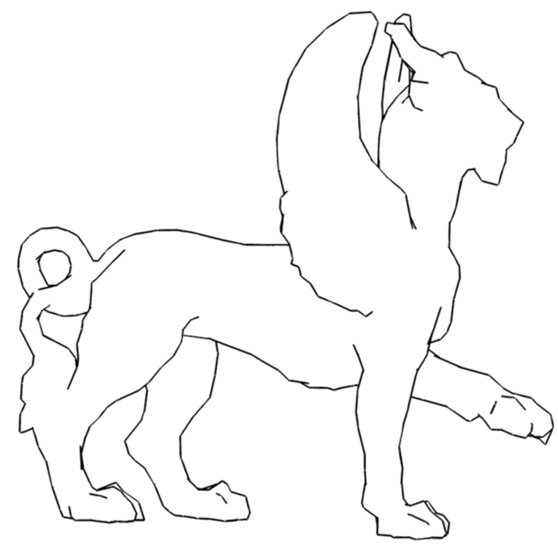
- Compute bounding box
- Compute Euler Characteristic
- Estimate surface curvature
- Line description for conveying surface shape
- Morse function and surface topology--Reeb graph
- Scalar field topology--Morse-Smale complex

Surface Curvature Estimation

By Prof. Eugene Zhang

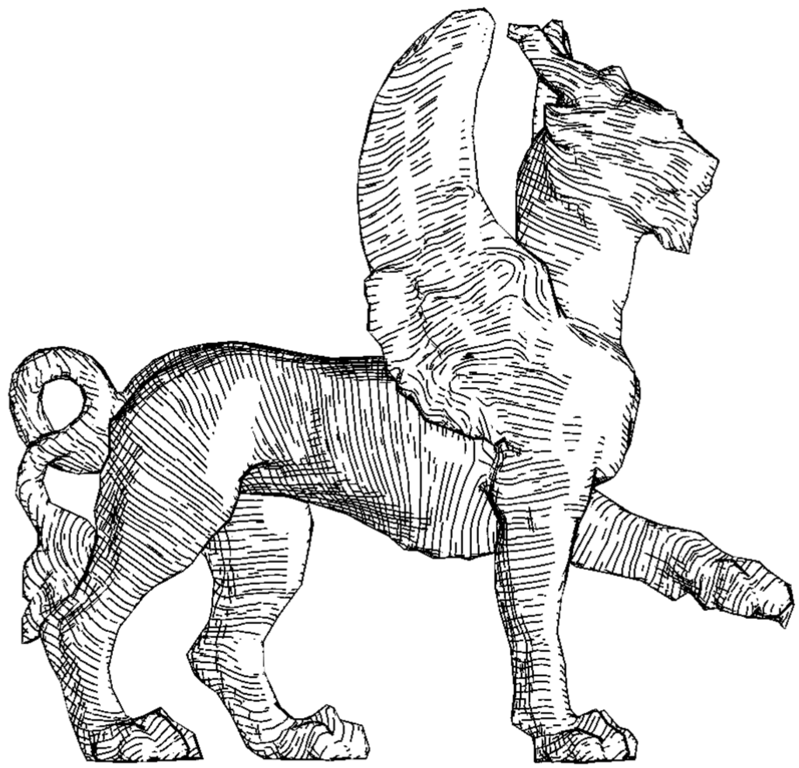
Shape Visualization

- How to convey shape with a few lines? What lines should be drawn?



Shape Visualization

- Placing lines along the principle curvature direction is best at illustrating the shape of an object



Curvature

- Curve
- Surface

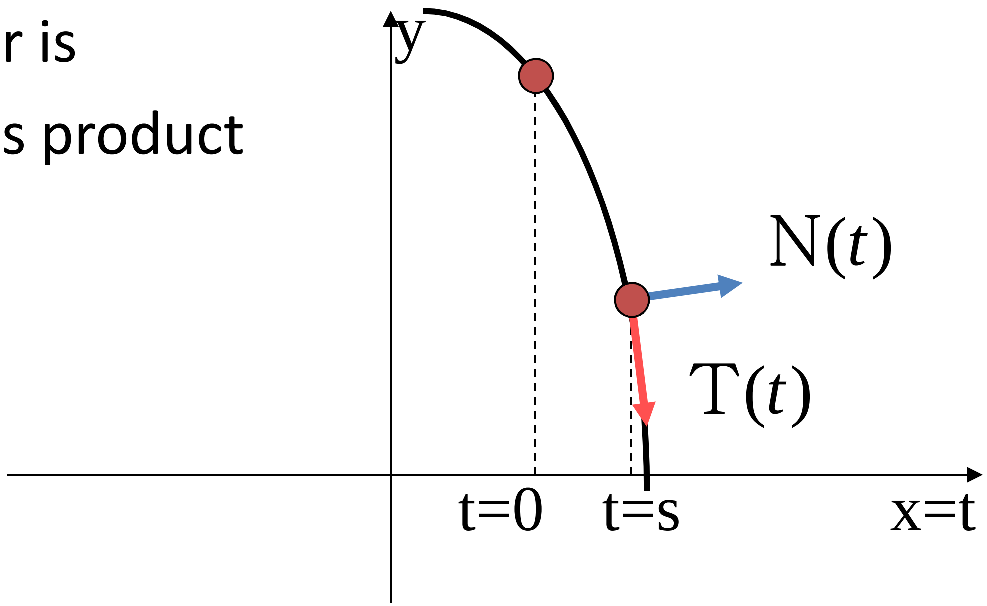
Curvature of a Planar Curve

A Planar Curve

- Given a 2D curve $\mathrm{r}(t) = (x(t), y(t))$
  - The unit tangent vector is defined as

$$\mathrm{T}(t) = \frac{\mathrm{r}'(t)}{|\mathrm{r}'(t)|} = \frac{(x'(t) \quad y'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

  - The unit normal vector is
  - counterclockwise cross product



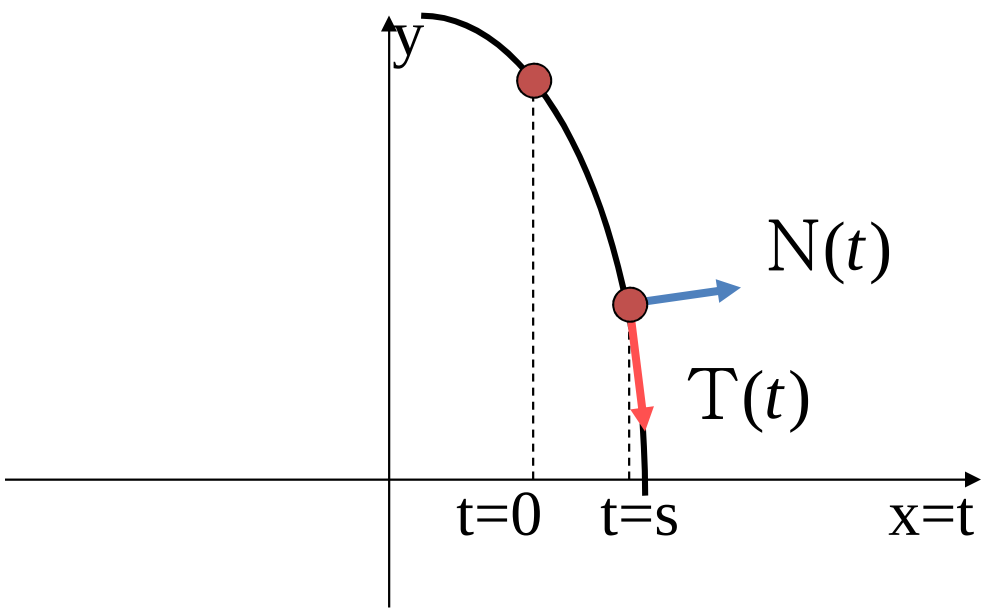
Arc Length

- The arc length of $r(t) = (x(t), y(t))$ is

$$s(u) = \int_{t=0}^{u} |r'(t)| dt = \int_{t=0}^{u} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

- and

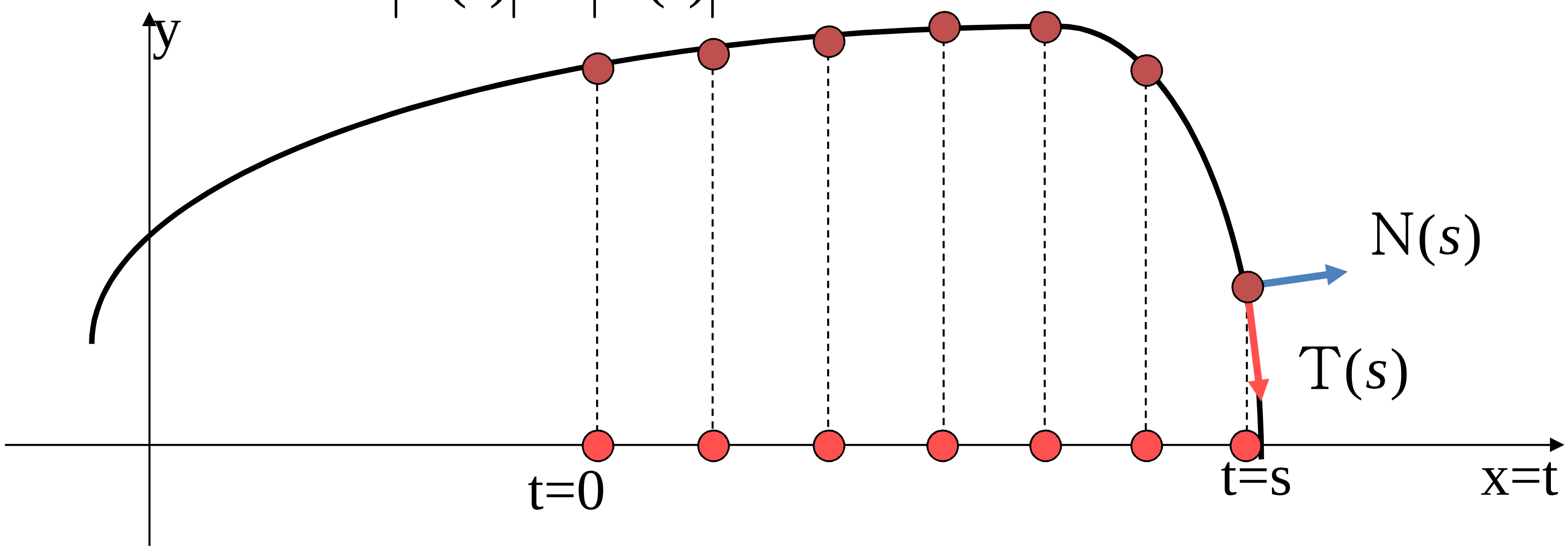
$$s'(t) = |r'(t)|$$



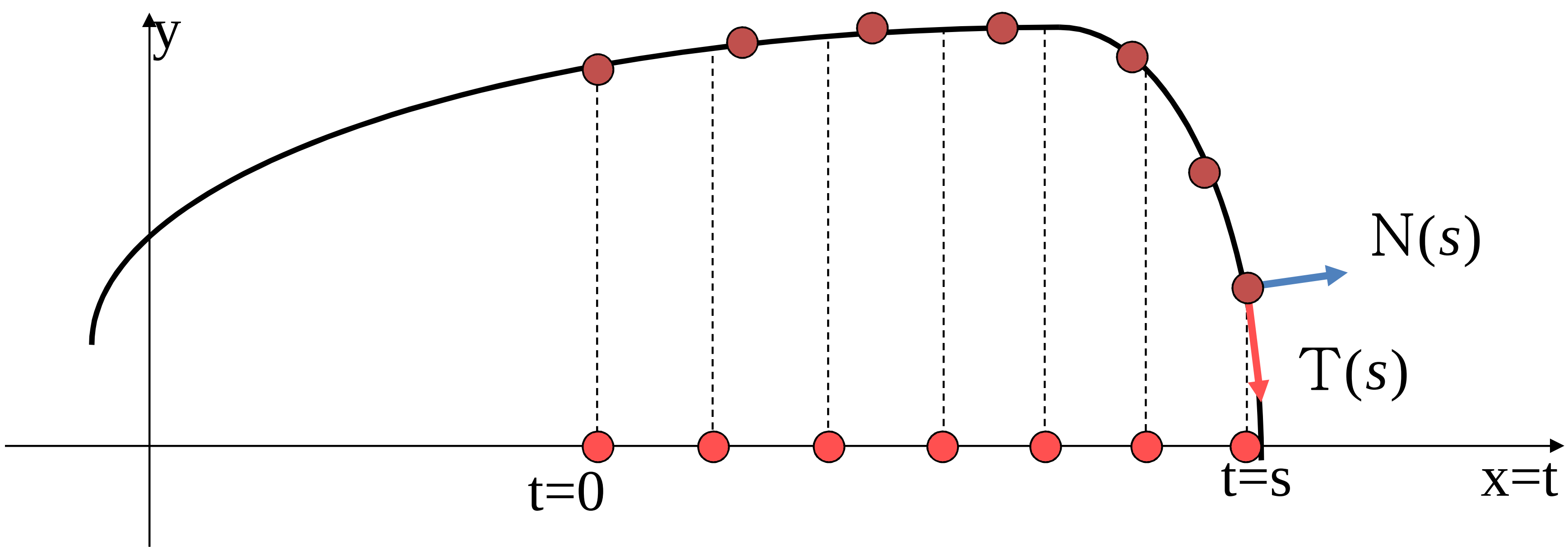
Arc Length Based Re-parameterization

- The curve $\mathrm{r}(t) = (x(t), y(t))$ becomes $\mathrm{r}(s) = (x(s), y(s))$
- and we have

$$|r'(s)| = \frac{|r'(t)|}{|s'(t)|} = \frac{|r'(t)|}{|r'(t)|} = 1$$



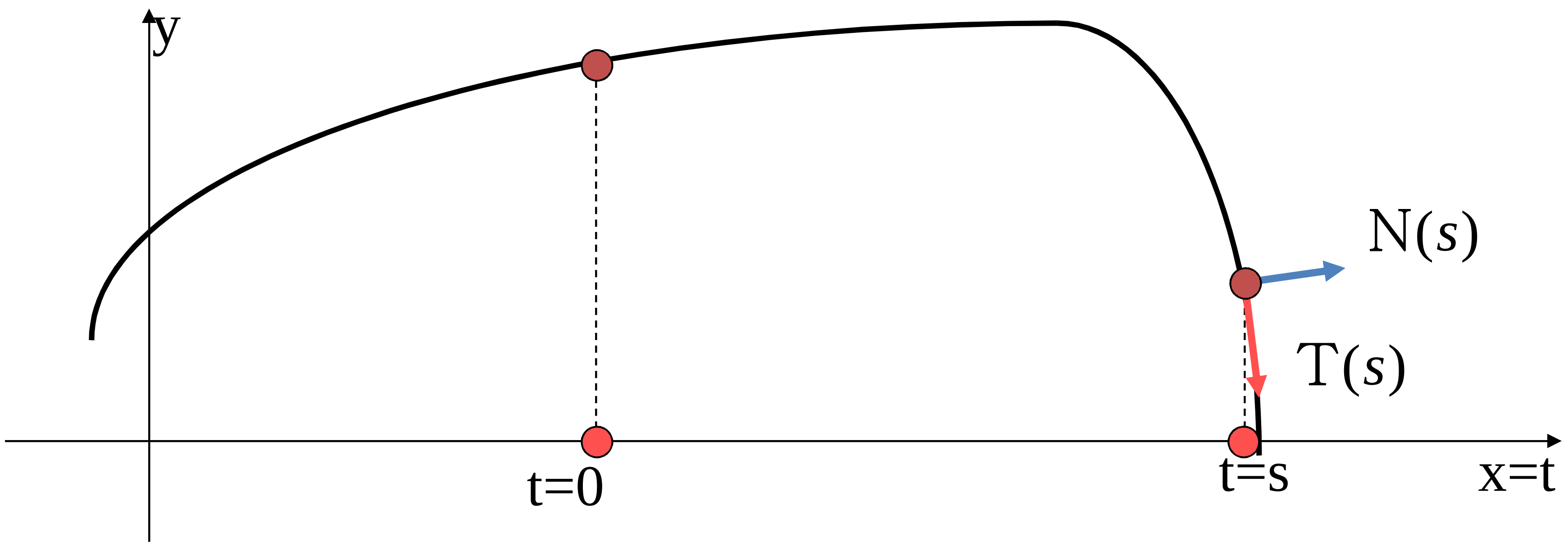
Arc Length Based Re-parameterization



Arc Length Based Re-parameterization

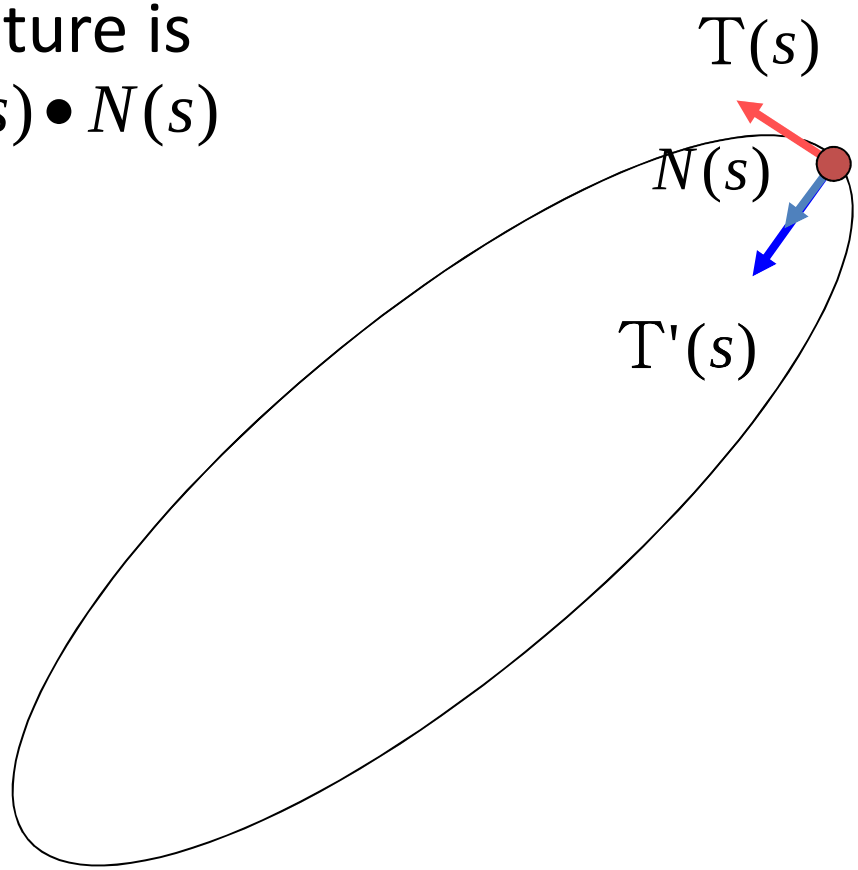
- The unit tangent vector is

$$T(s) = \frac{r'(s)}{\left|r'(s)\right|} = r'(s)$$

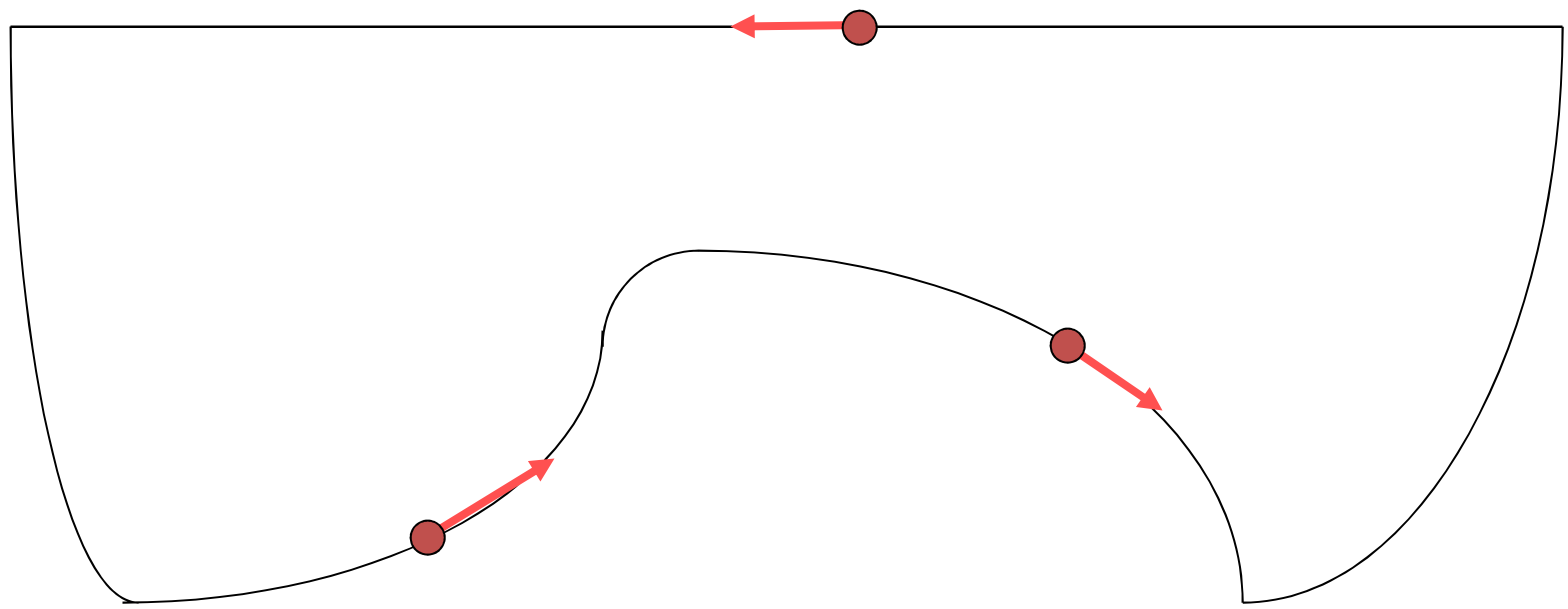


Curvature

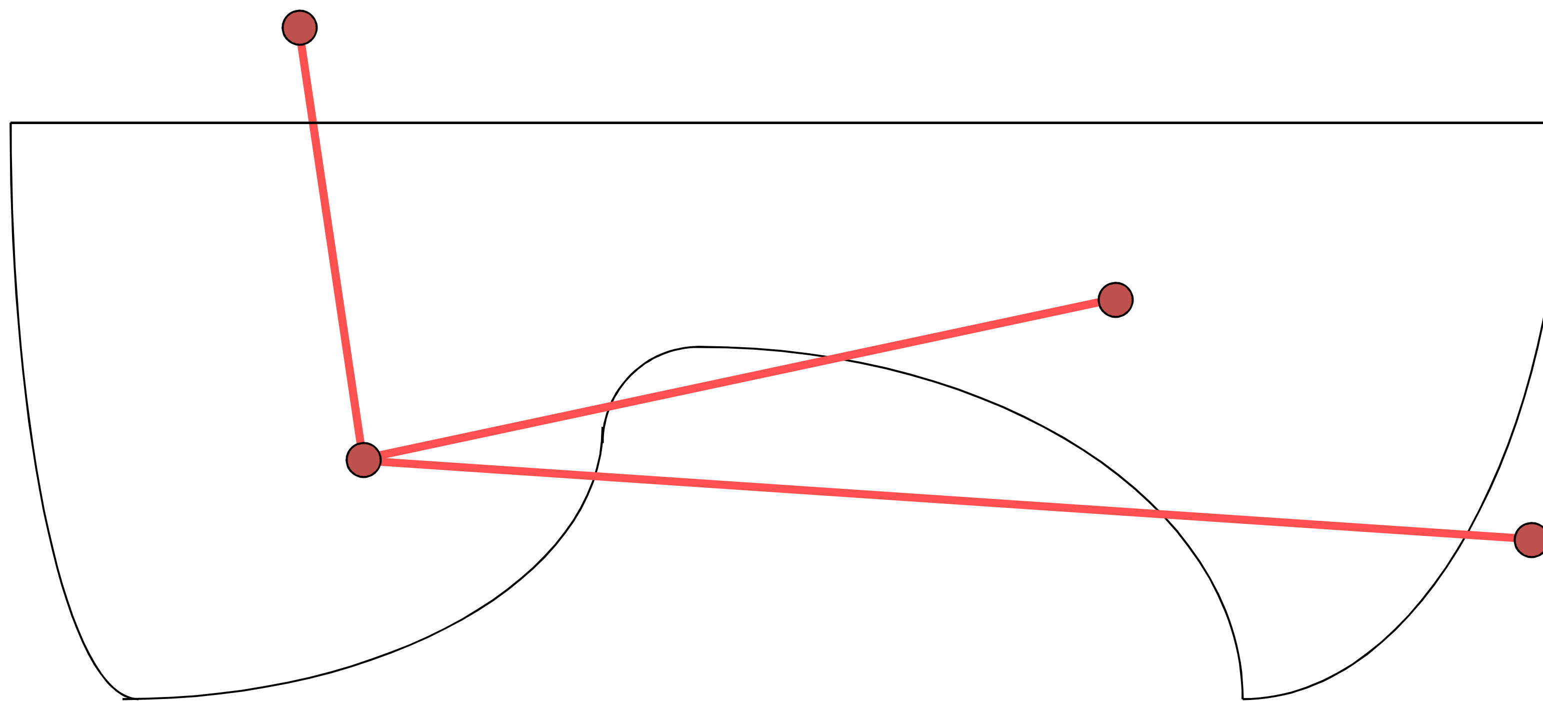
- The curvature is $\kappa(s) = T'(s) \bullet N(s)$



Signed Curvature



Topology of Curves



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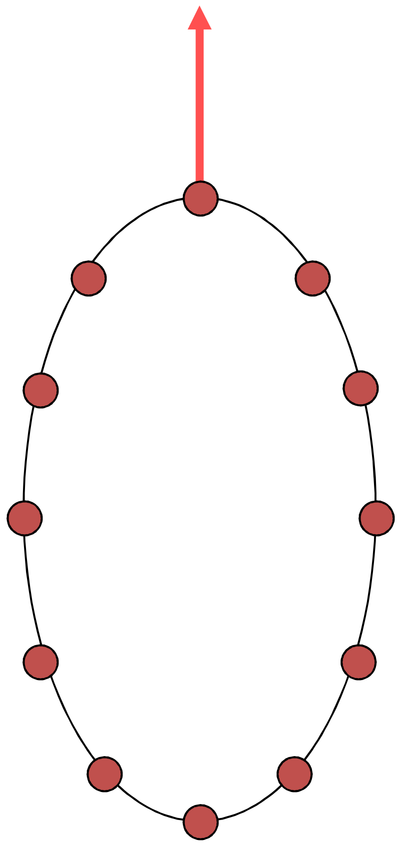
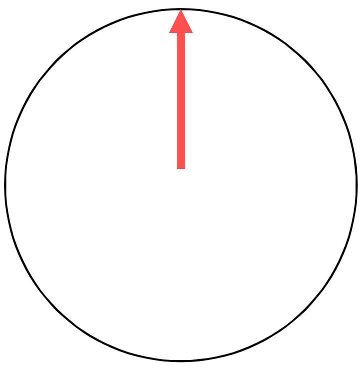
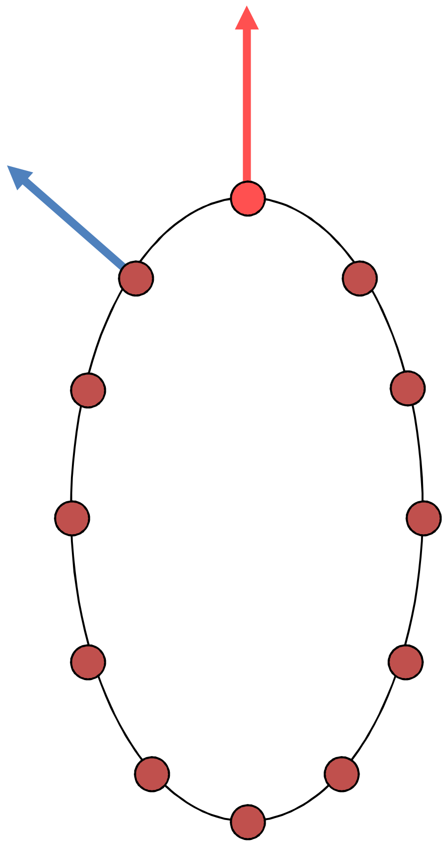
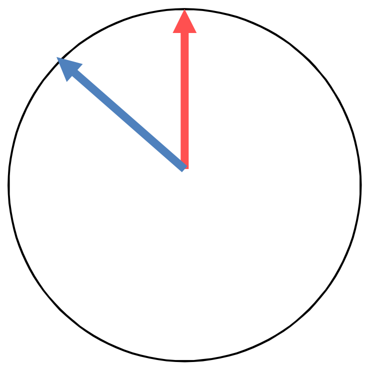




Gauss Circle

Topology of Curves





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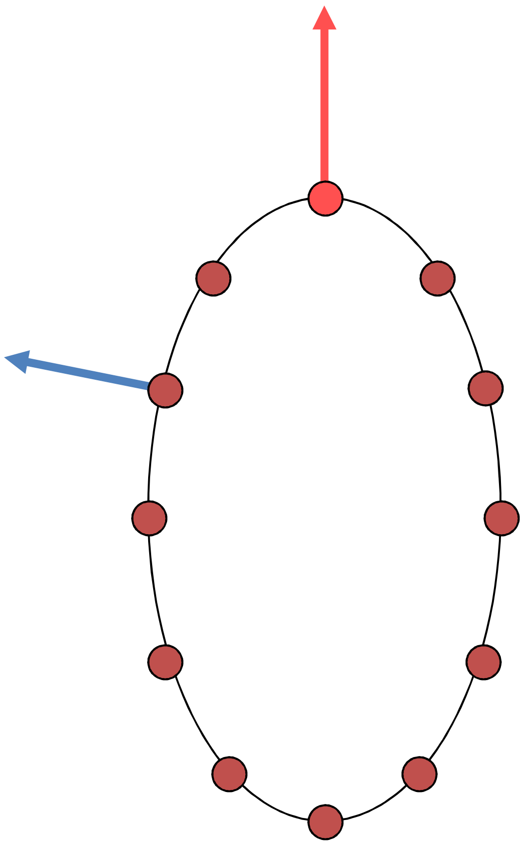

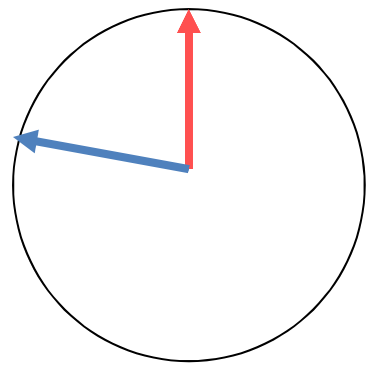

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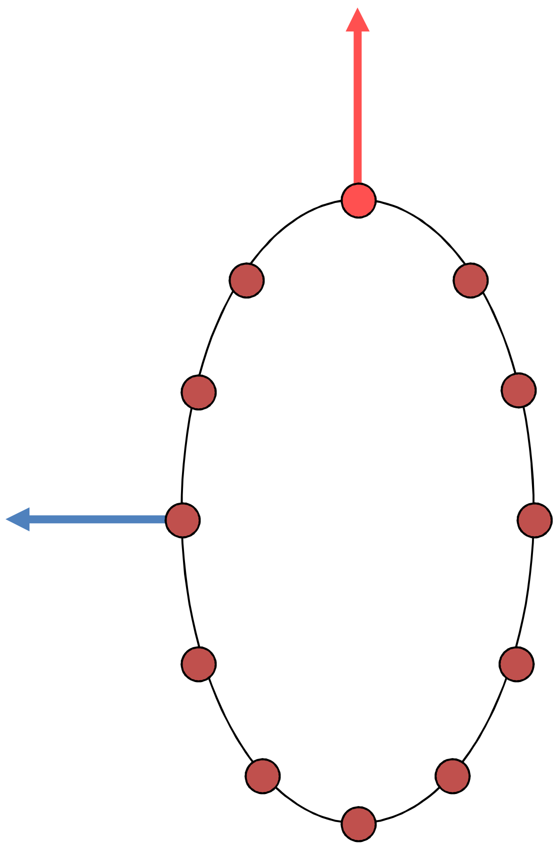

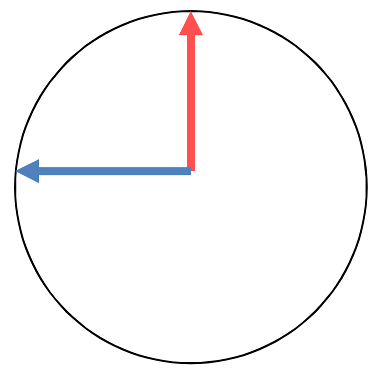

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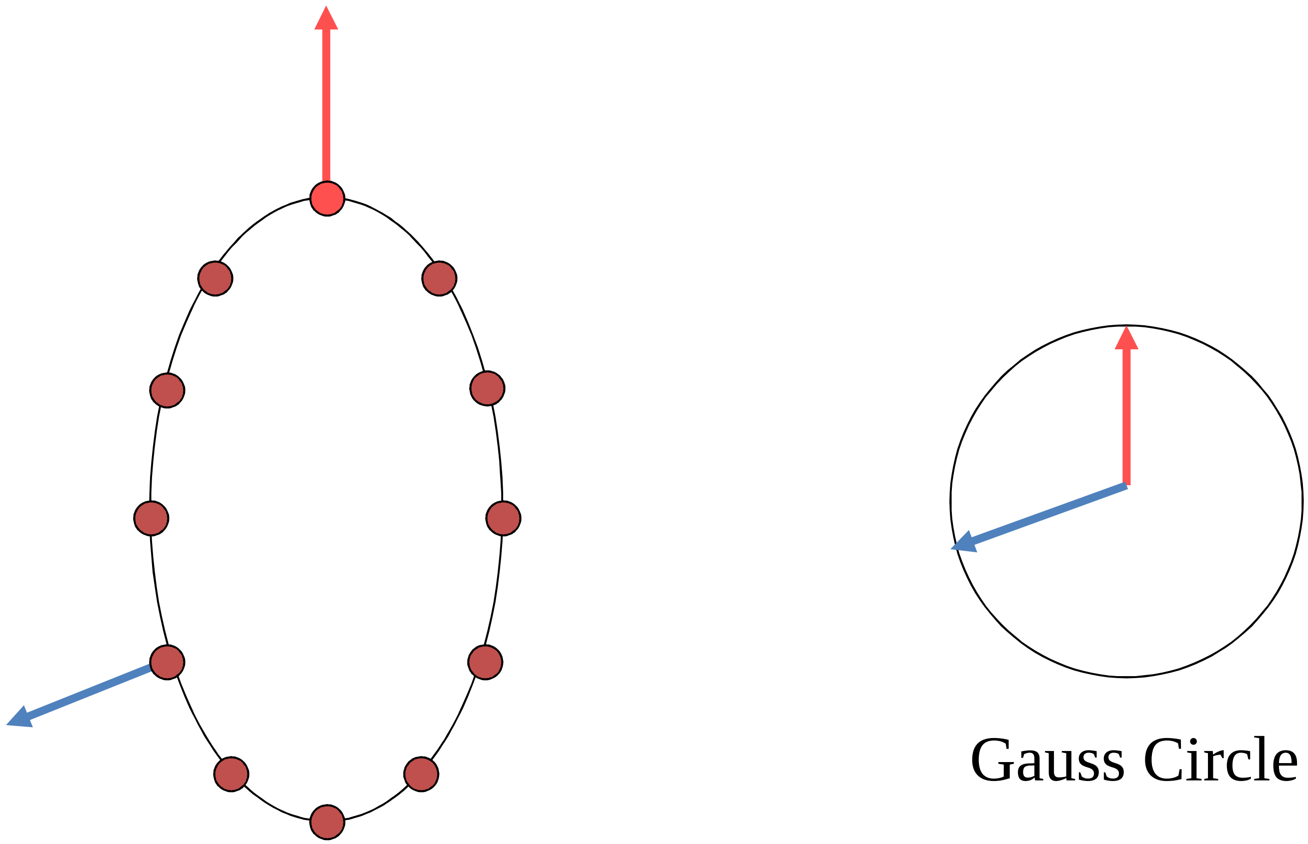

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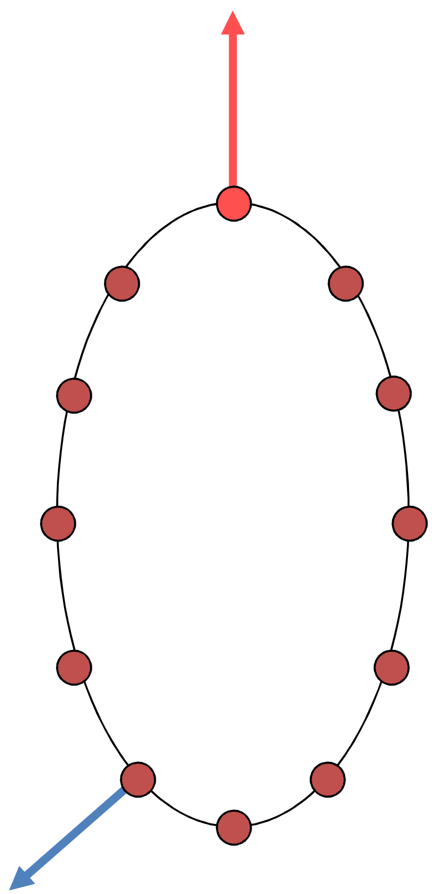
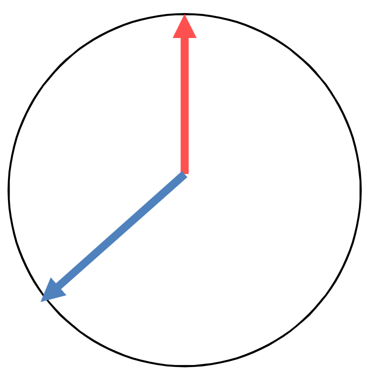
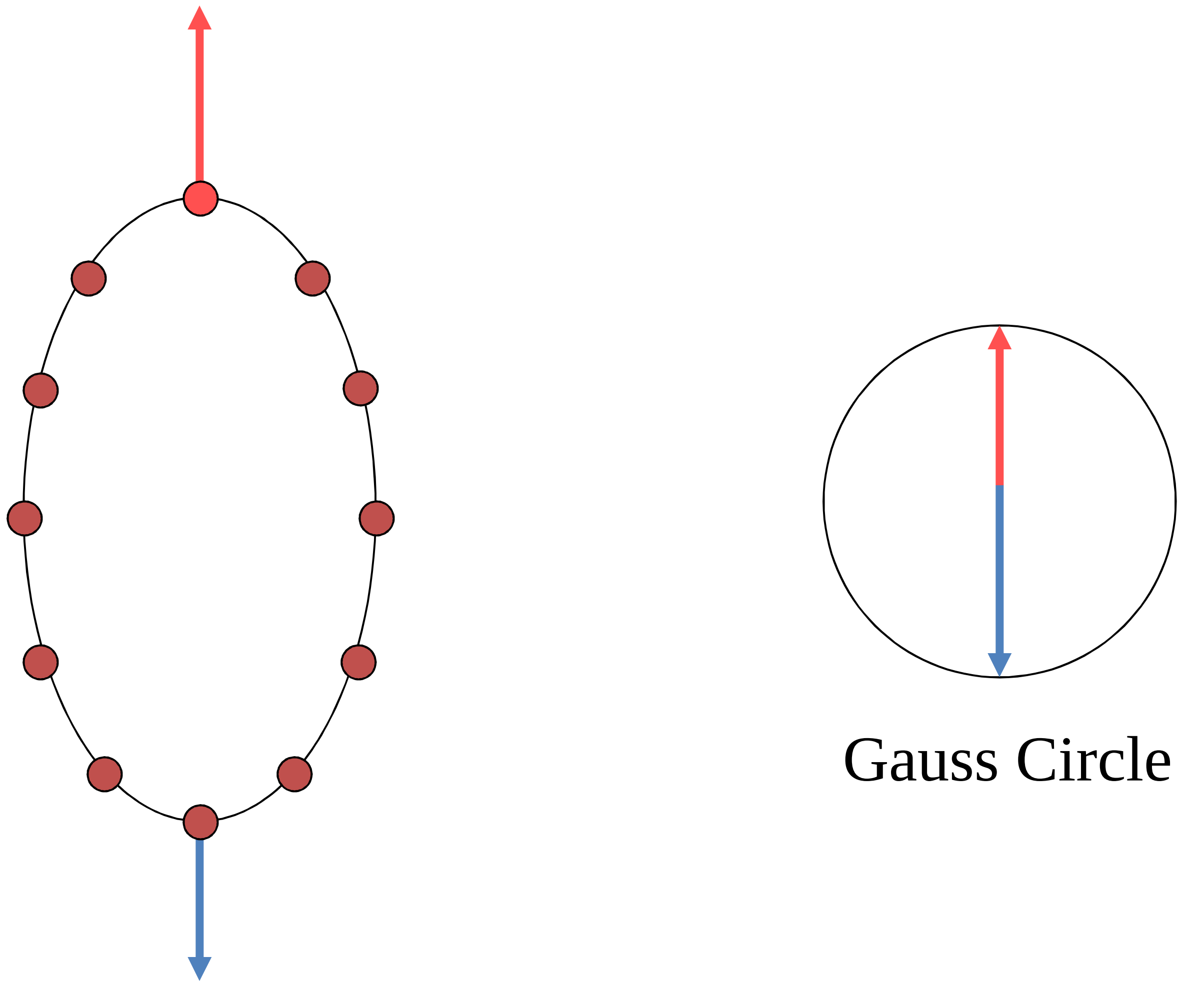
Topology of Curves





Gauss Circle

Topology of Curves



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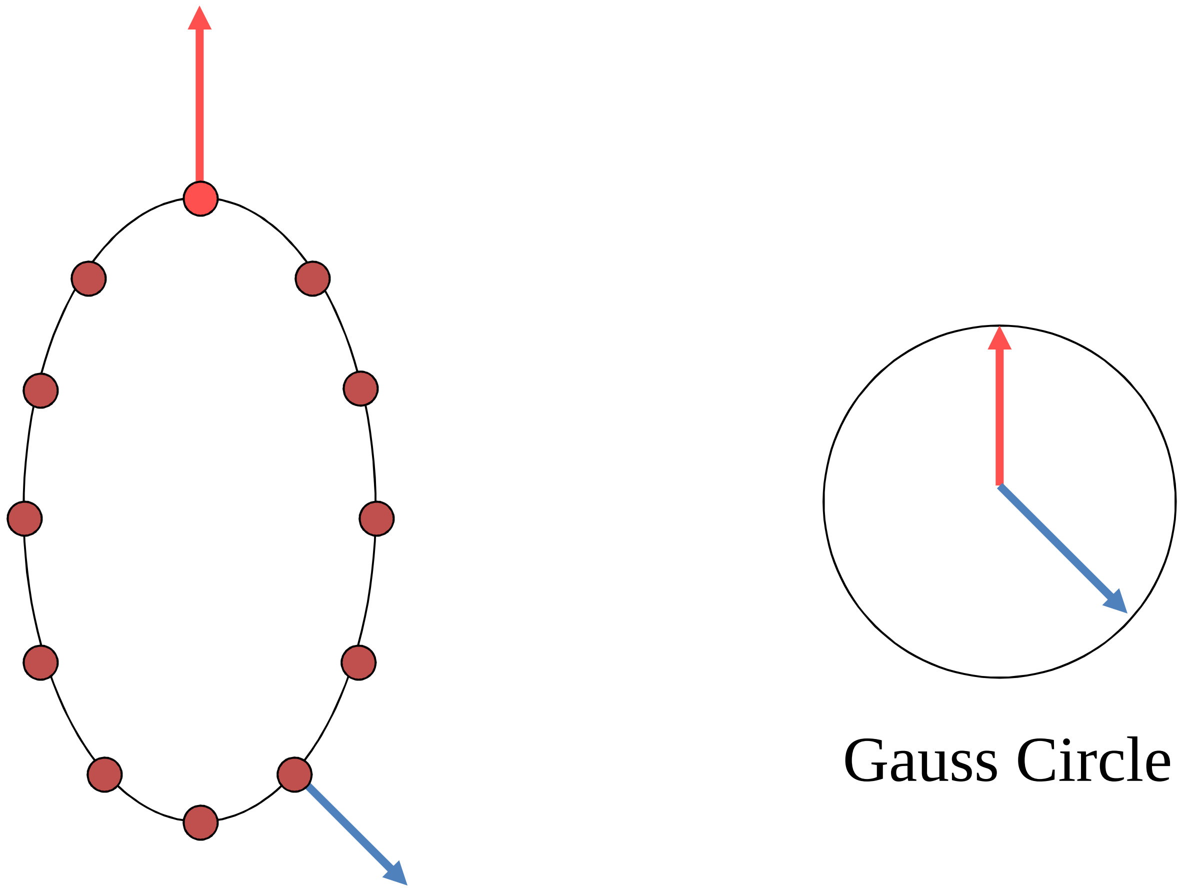

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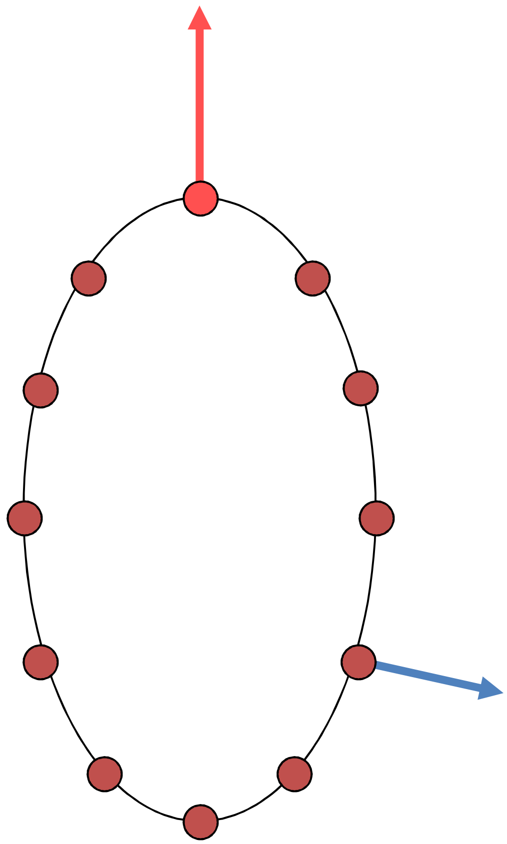
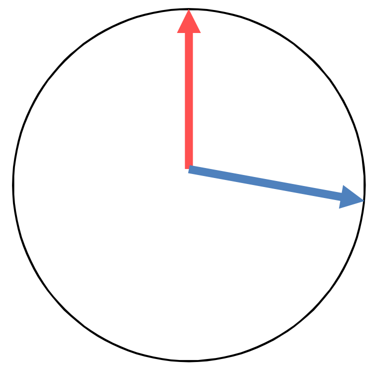
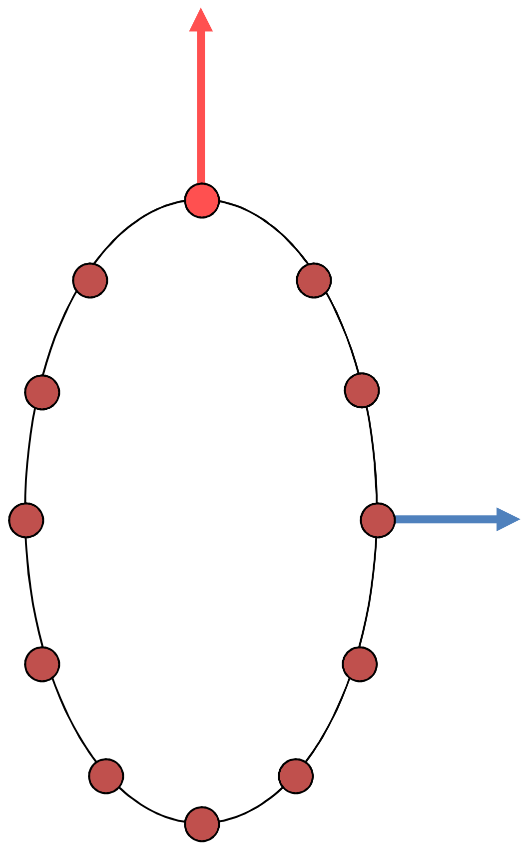
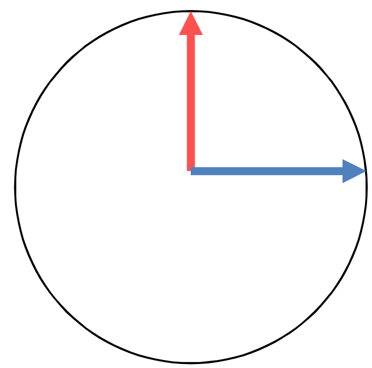




Gauss Circle

Topology of Curves





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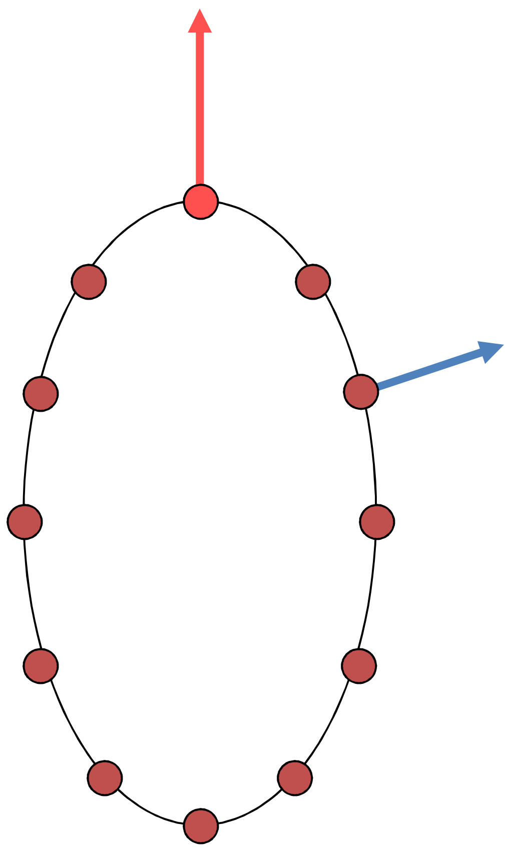

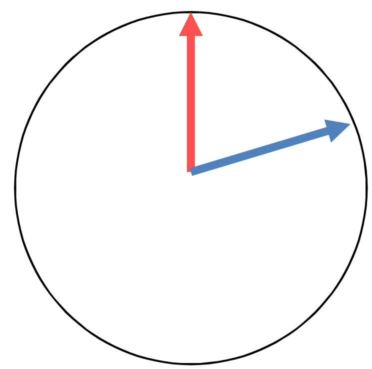

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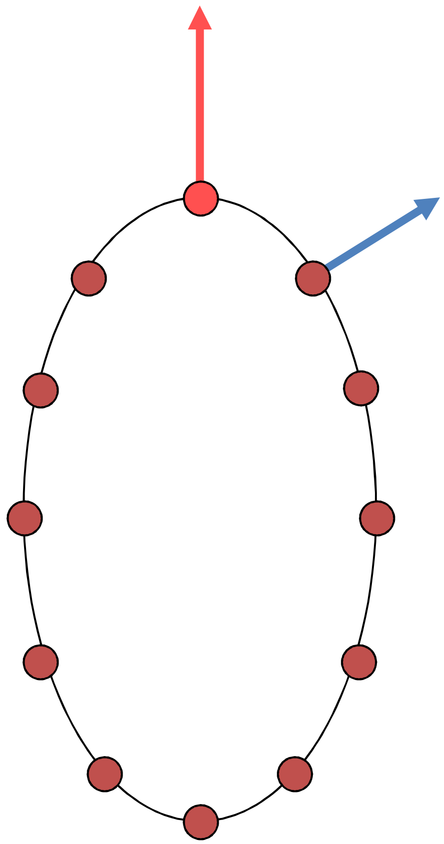

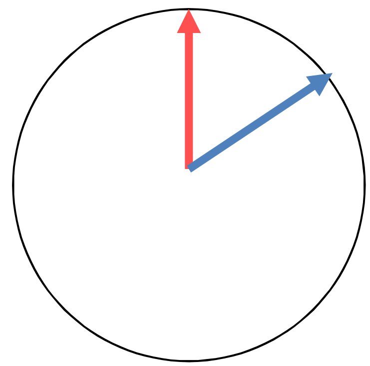

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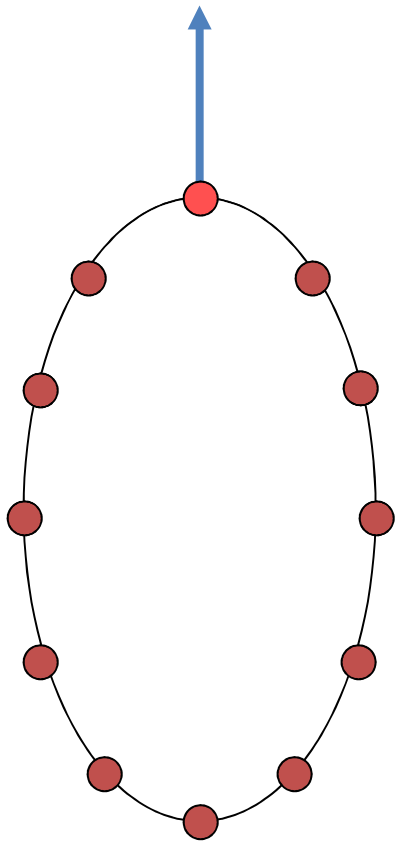

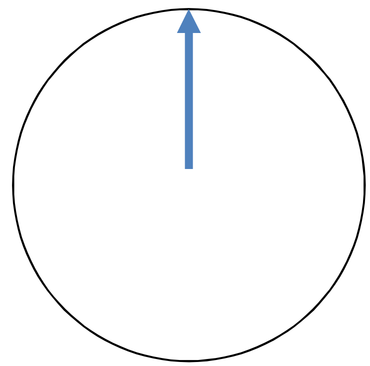

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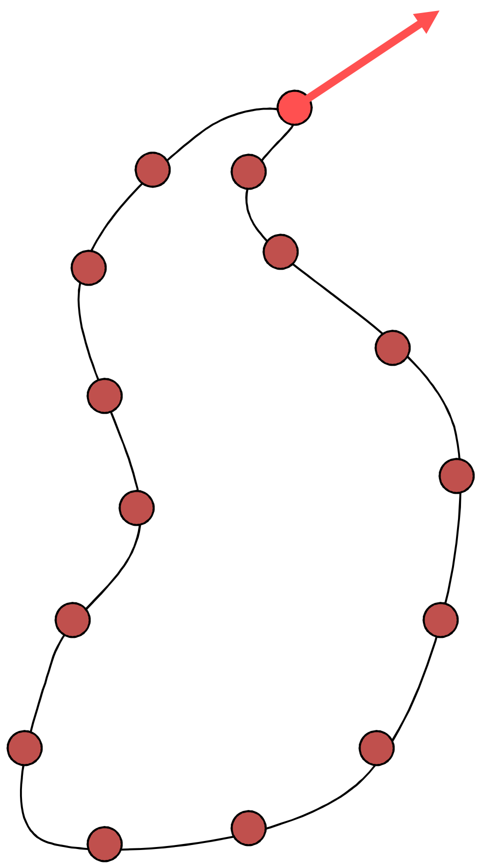

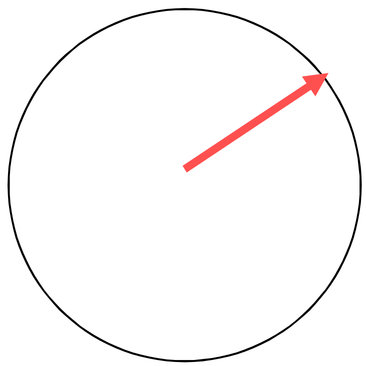

Gauss Circle

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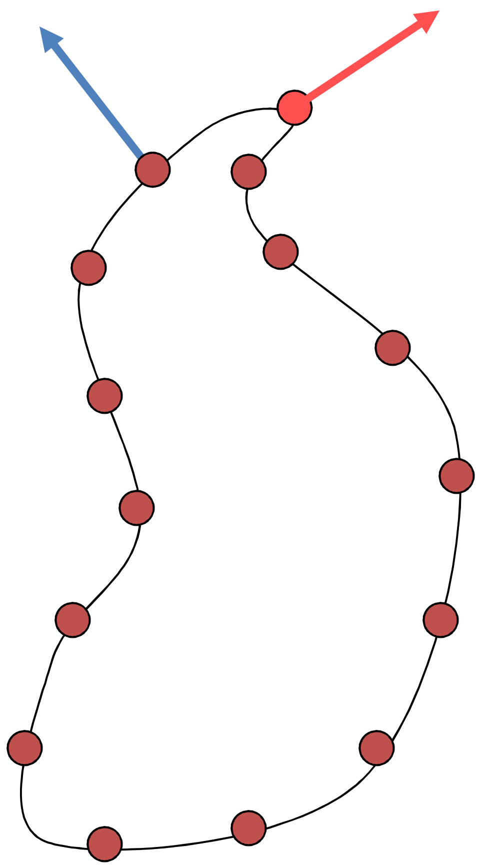

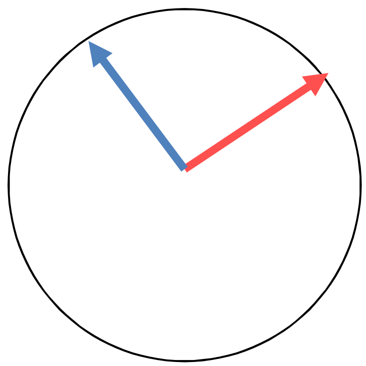

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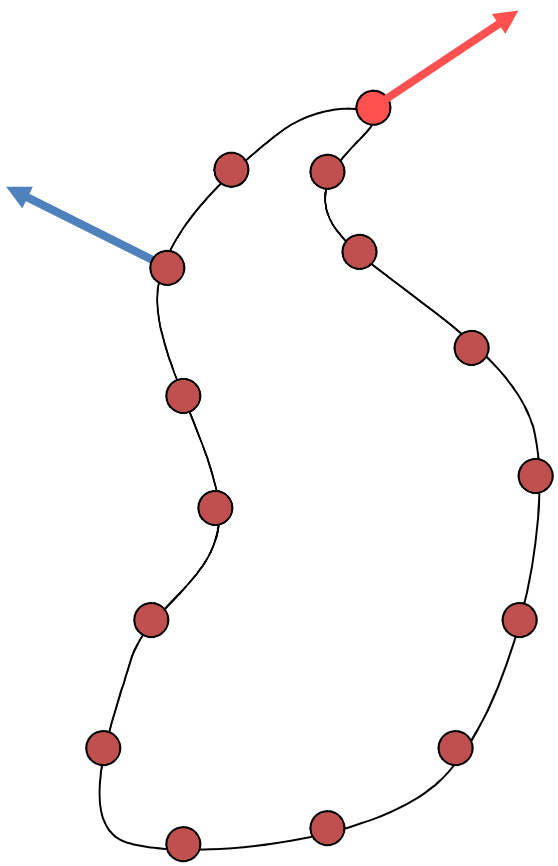

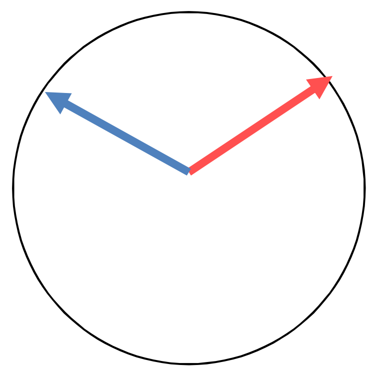

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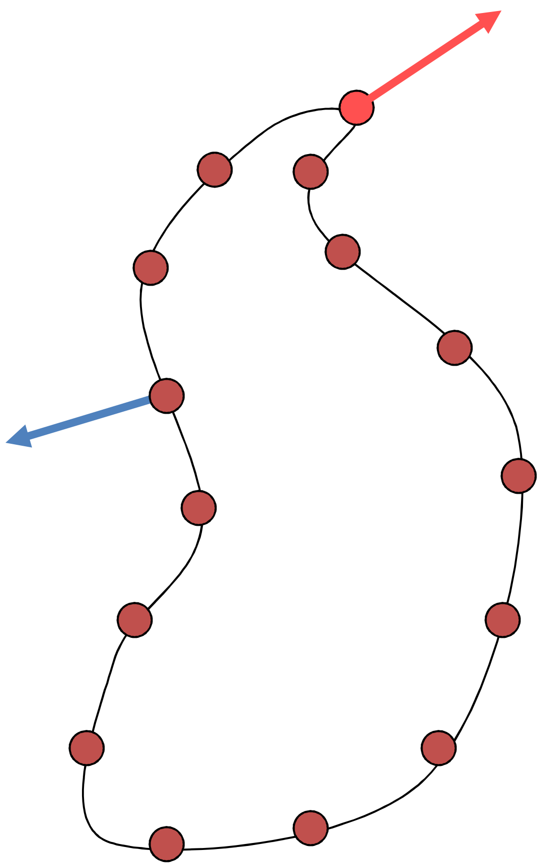

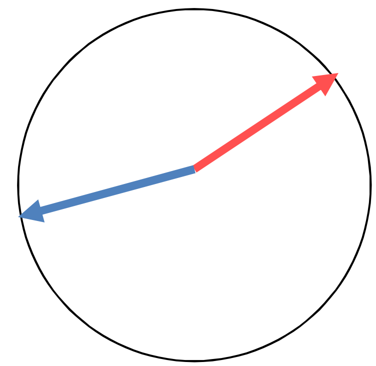

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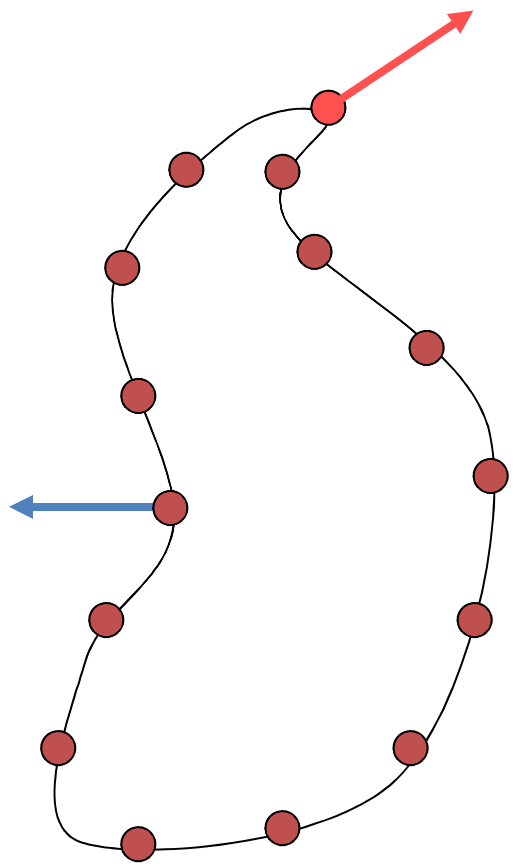

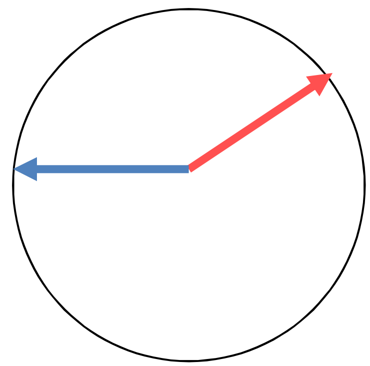

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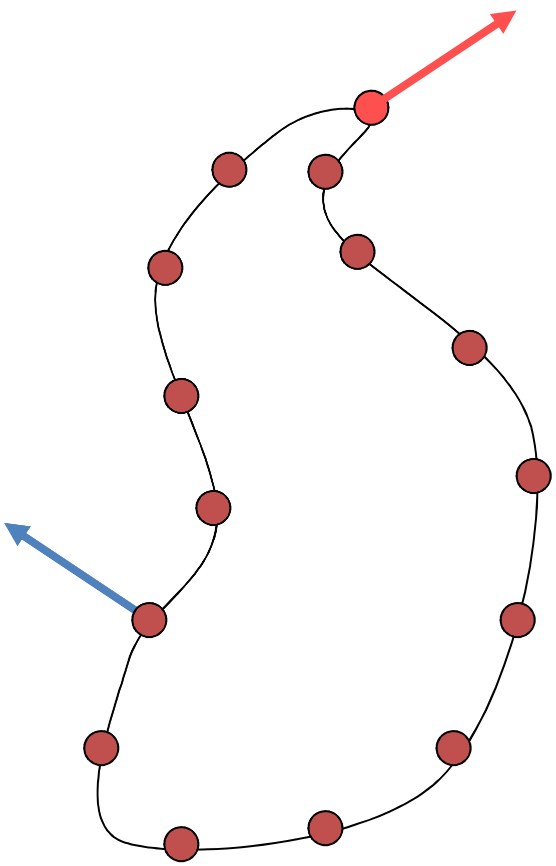

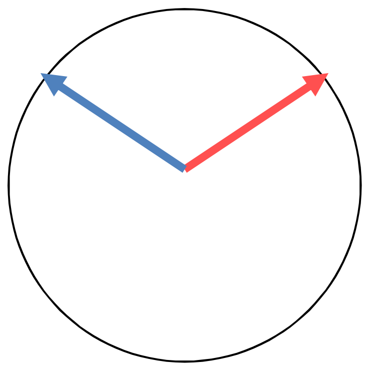

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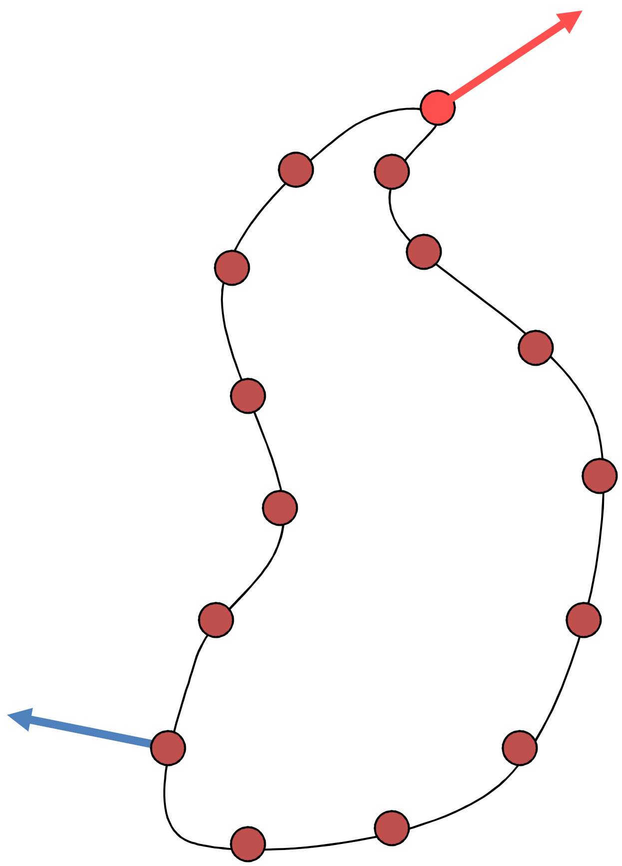

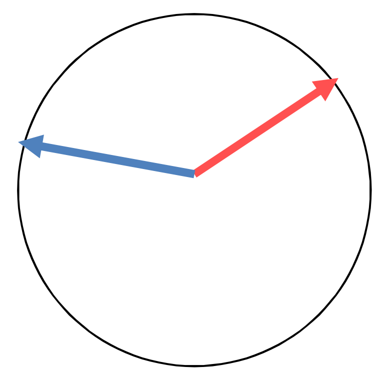

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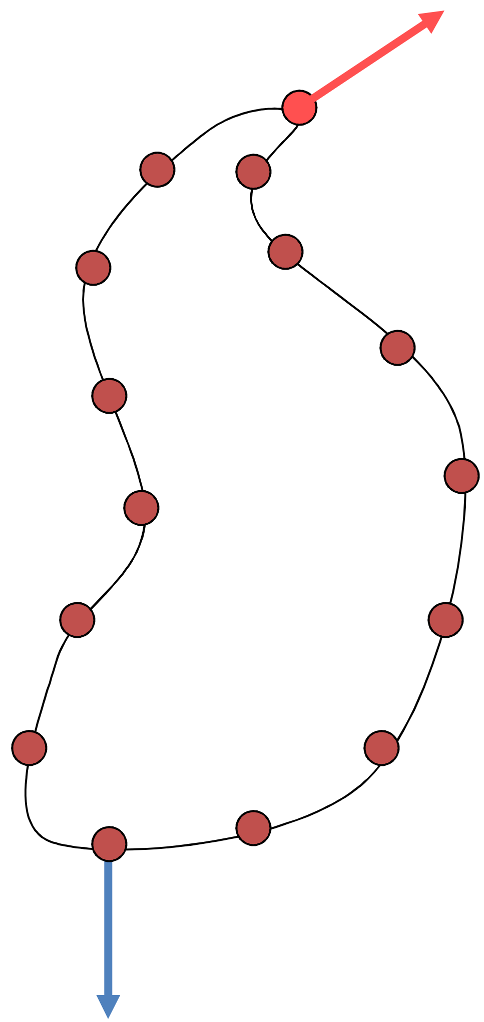
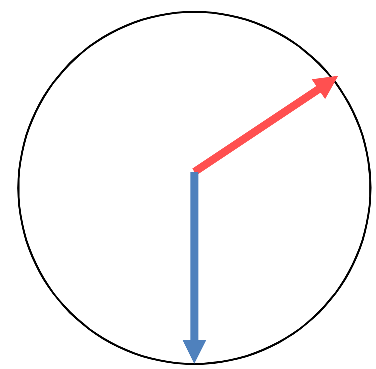
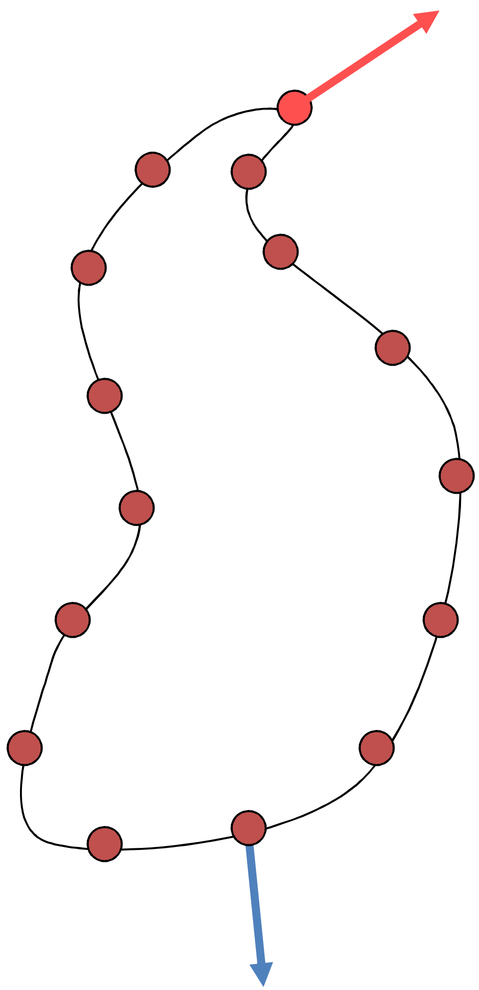
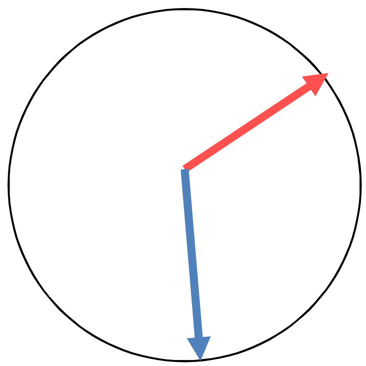




Gauss Circle

Topology of Curves





Gauss Circle

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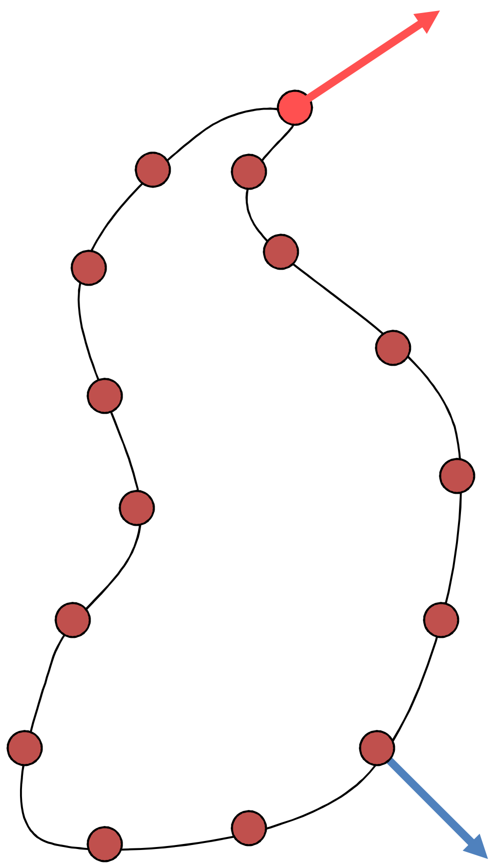

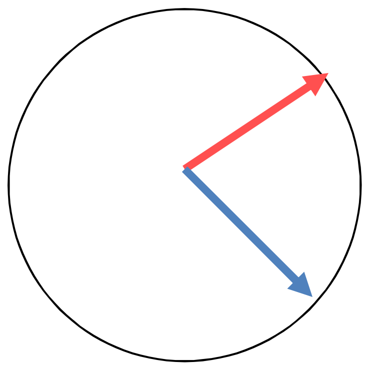

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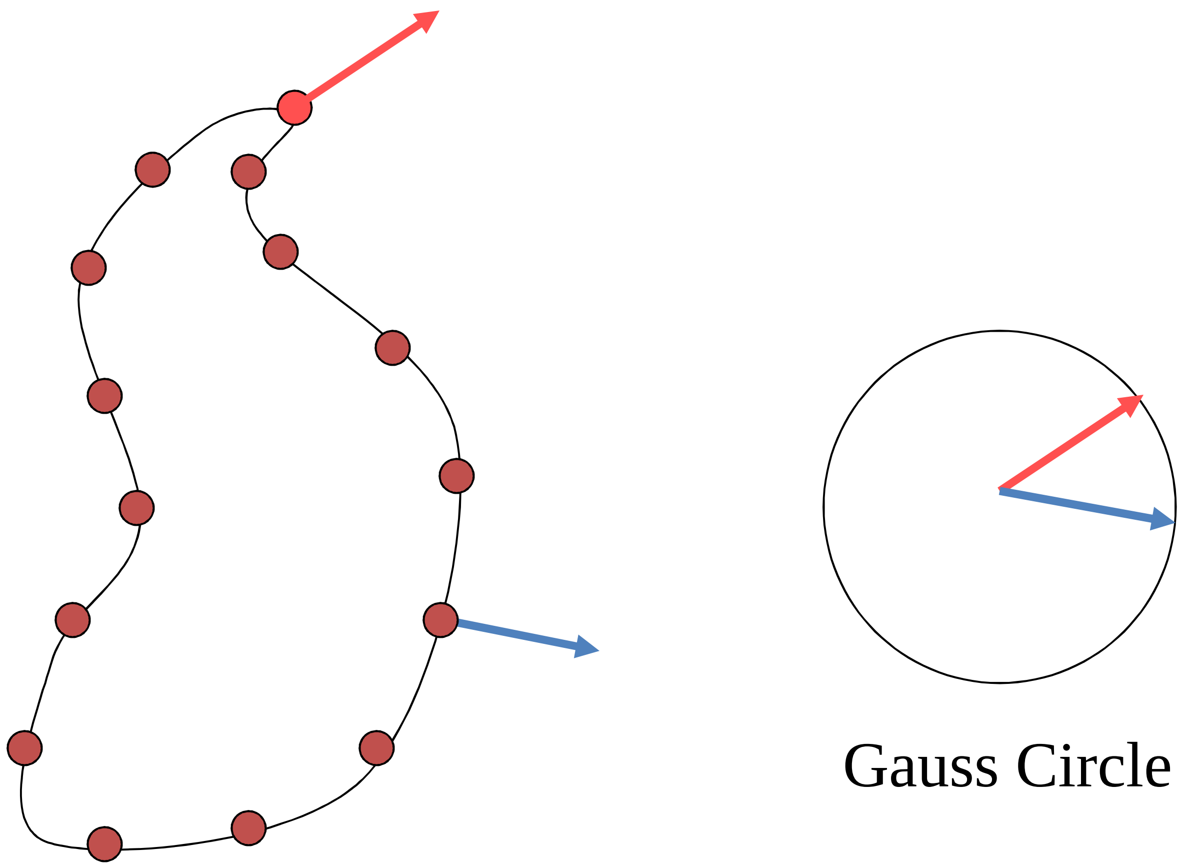

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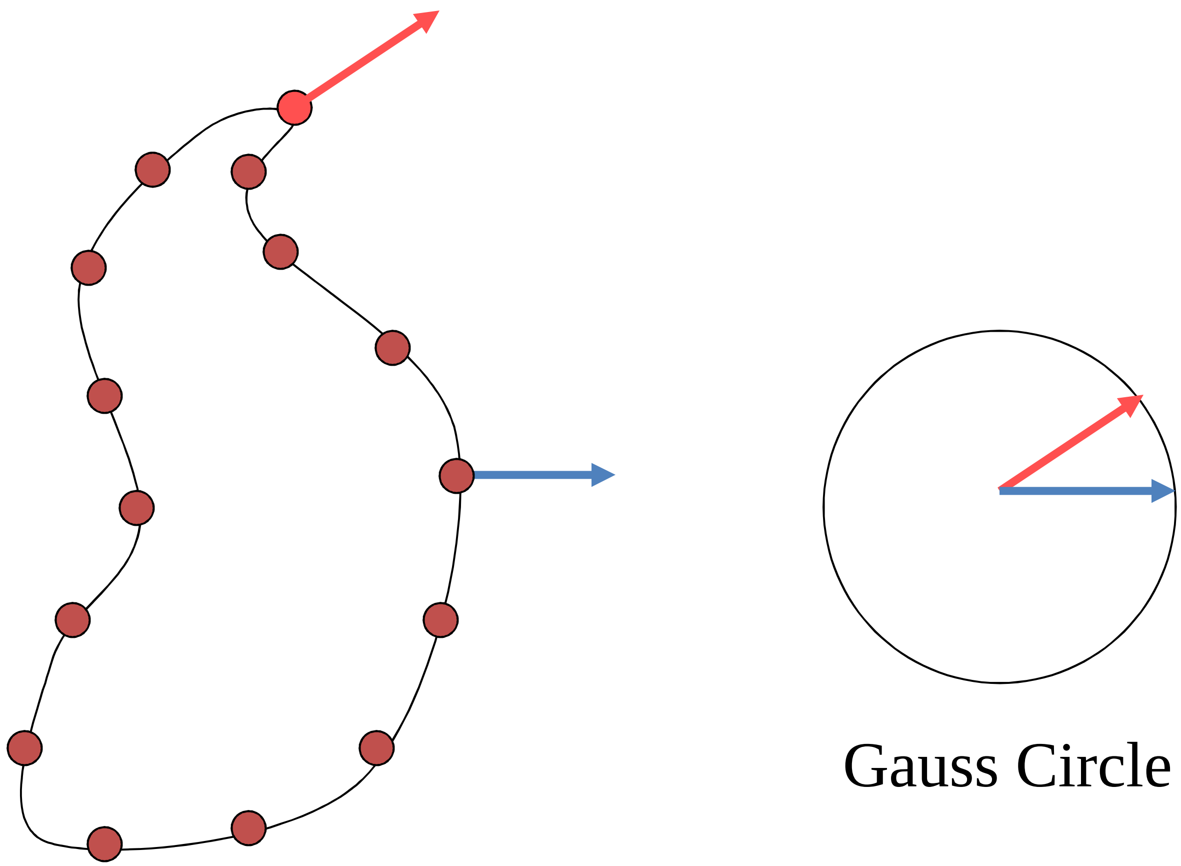

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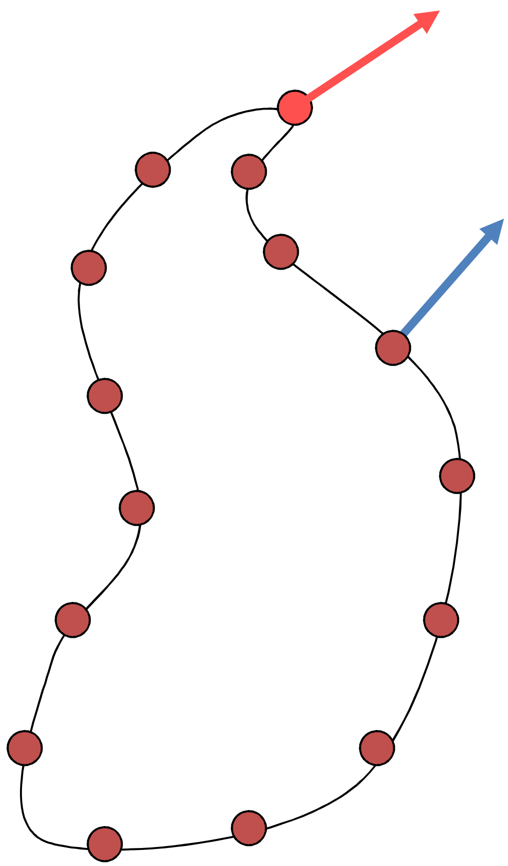

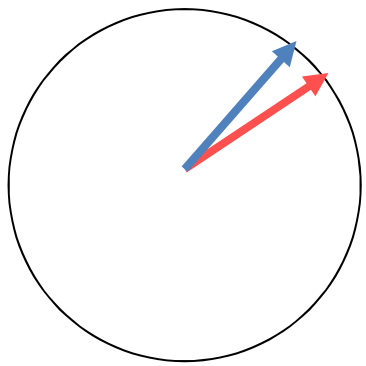

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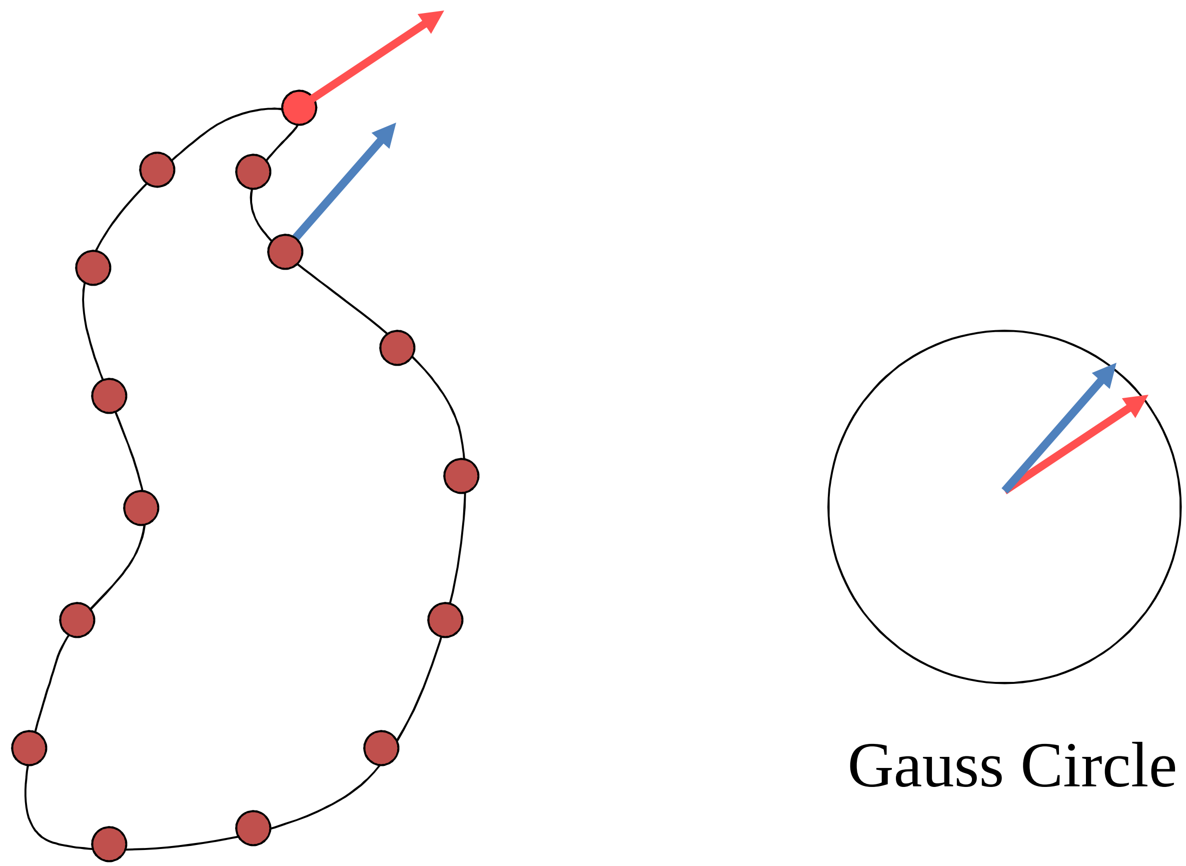

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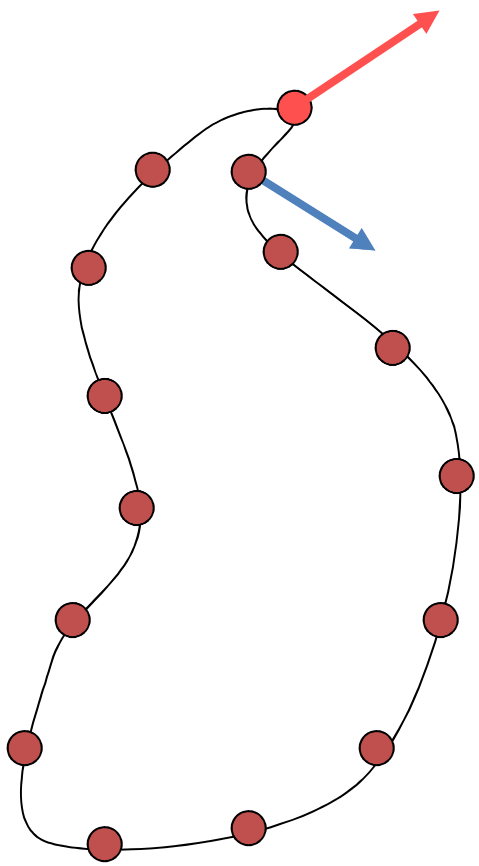

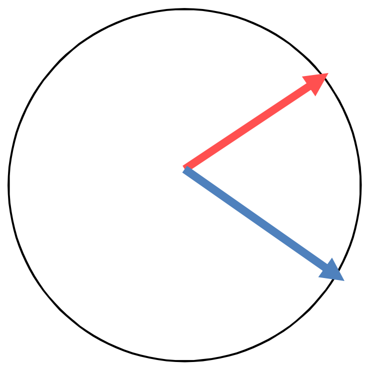

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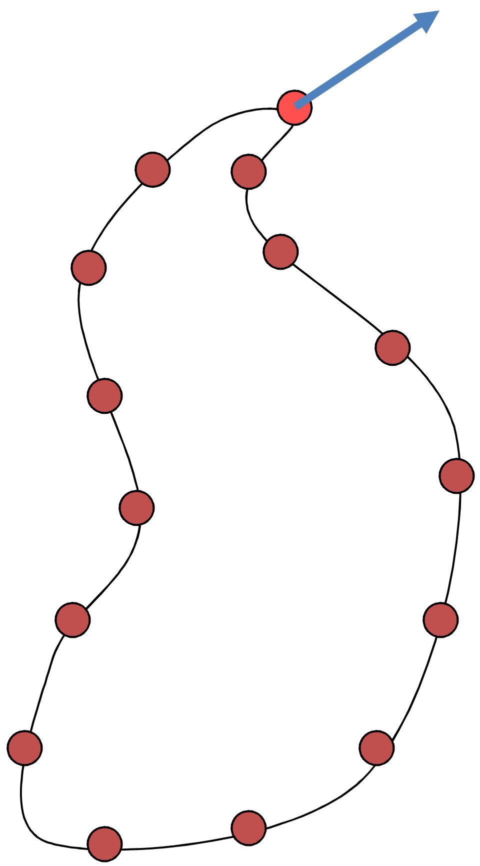
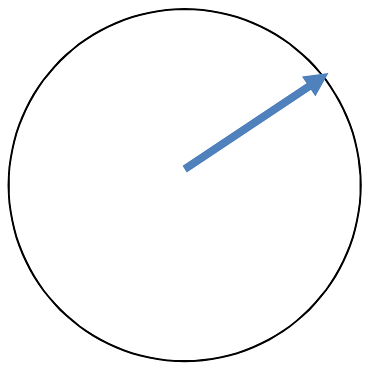




Gauss Circle

Conclusion

- Recall $\kappa(s)$ measures how fast unit tangent vectors change directions.
- Therefore, it also measures how fast unit normal vectors change directions, counting orientation
- The total curvature along a closed curve is 2π
- This shows that the total curvature of a curve is a topological quantity

Curvature of a Surface

Surface Curvature

- Use curvature of curves to define curvature of a surface
- For a point **P** on the surface
  - For every unit tangent vector V at **P**
    - Construct the plane that contains P and is parallel to V and N
    - Find the intersection of the surface and this plane
    - Compute the curvature of the intersection curve (called normal curvature)

Surface Curvature

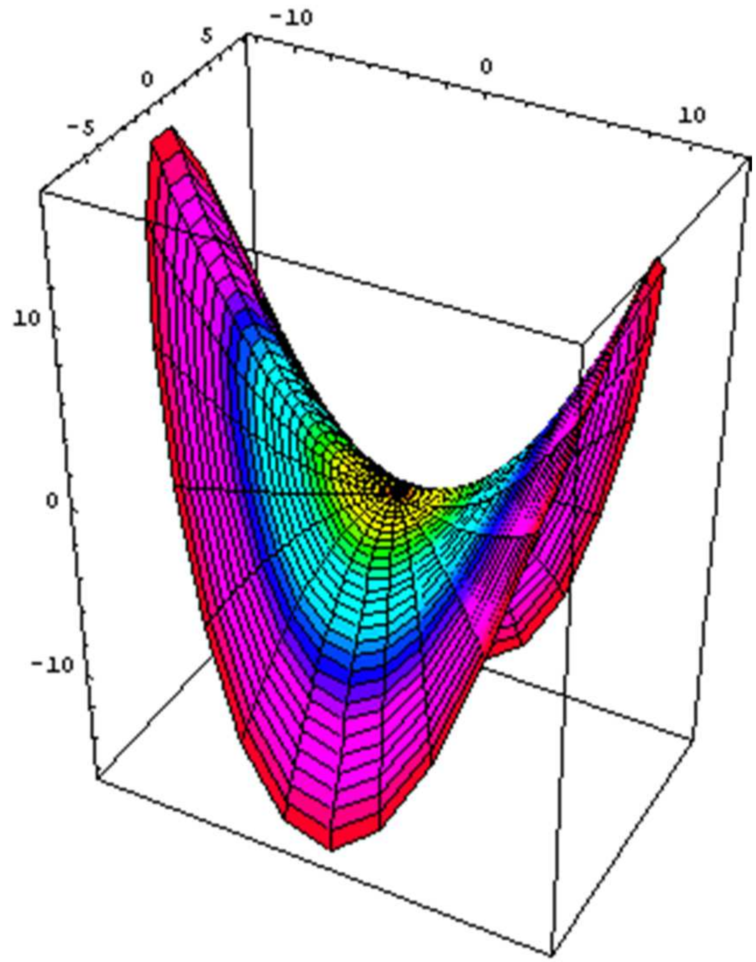


Image credit: http://www.lsus.edu/sc/math/rmabry/math223/quadricsurfaces/Images/index_gr_62.gif

Surface Curvature

- Let $t = aS_u + bS_v$ be a unit tangent vector
- The normal curvature in the direction *t* is:

$$S_{tt} \bullet N = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} S_{uu} \bullet N & S_{uv} \bullet N \\ S_{vu} \bullet N & S_{vv} \bullet N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Tangent Space

- Classifying curves of a surface through a point p

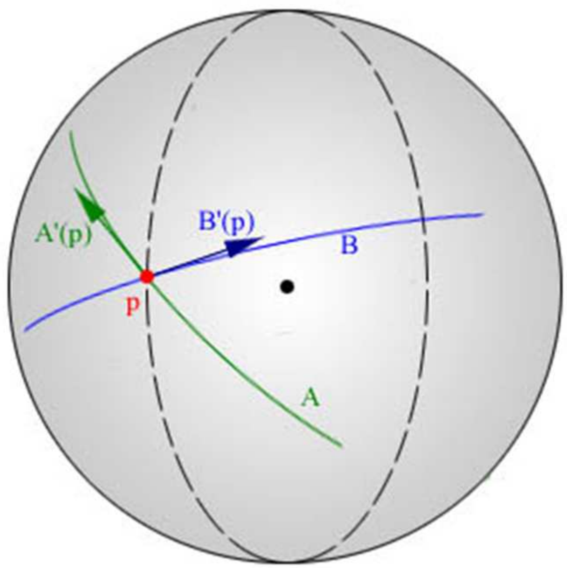


Image credit: www.peroxide.dk/.../tut10/pxdtut10.html

Tangent Space

- Two curves are equivalent if they are tangent to each other at p.

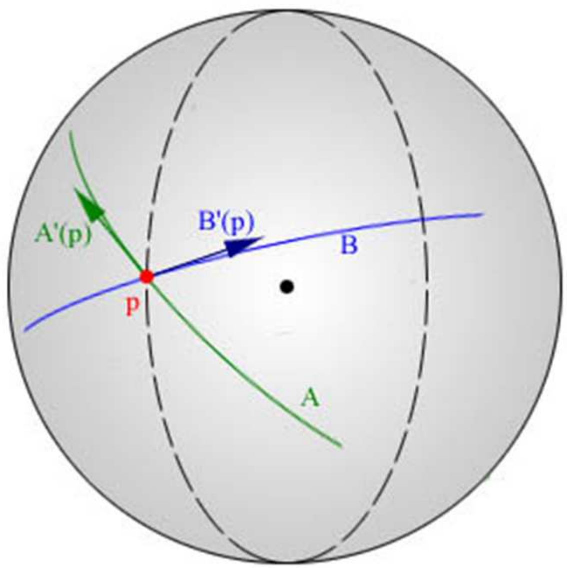


Image credit: www.peroxide.dk/.../tut10/pxdtut10.html

Tangent Space

- All different equivalent classes of curves form a line space: tangent space.

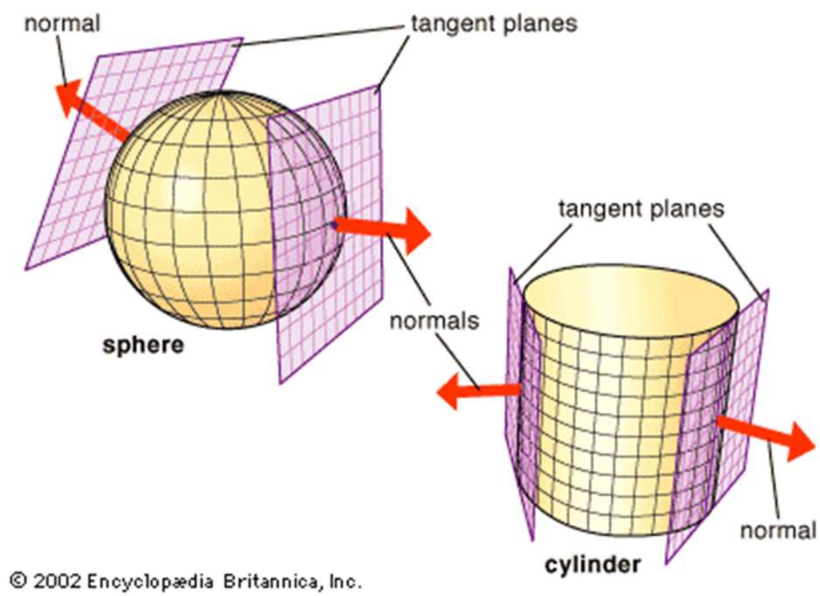


Image credit: **http://cache.eb.com/eb/image?id=70820&rendTypeId=4**

Tangent Space

- Each element in the tangent space, a vector, represents a class of equivalent curves.

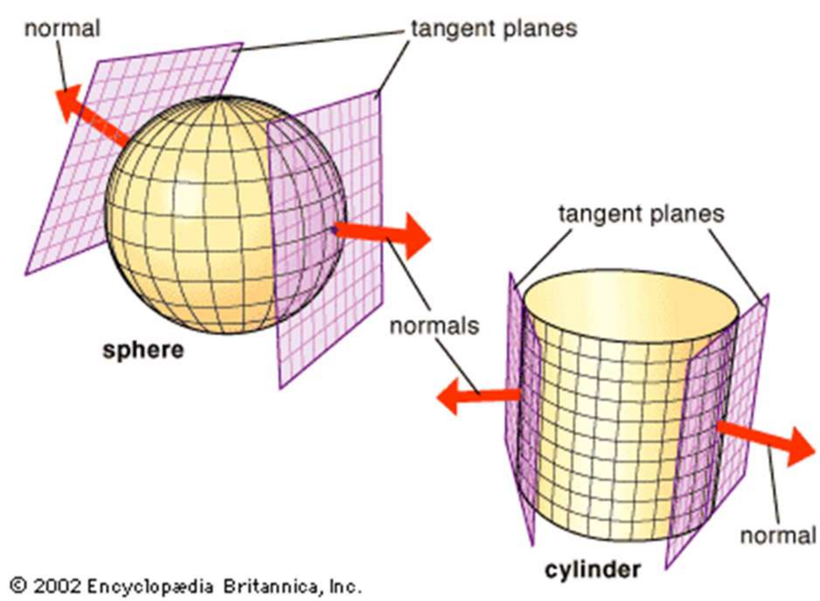


Image credit: **http://cache.eb.com/eb/image?id=70820&rendTypeId=4**

Tangent Space and Curvature

- All curves belonging to the same class have the same curvature at point p.

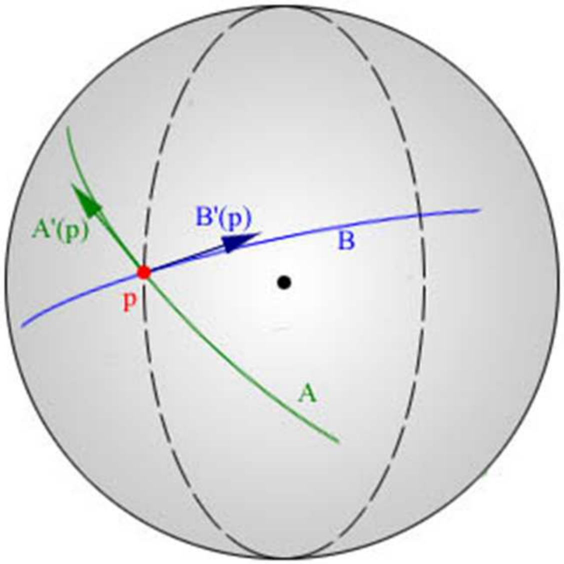


Image credit: www.peroxide.dk/.../tut10/pxdtut10.html

Tangent Space and Curvature

- Curvature depends only on the tangent vector.

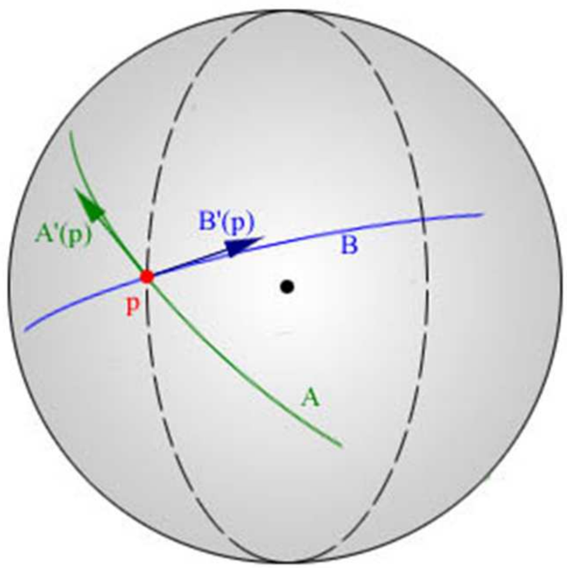


Image credit: www.peroxide.dk/.../tut10/pxdtut10.html

Tangent Space and Curvature

- A tangent vector can be represented by

  $t = aS_u + bS_v$ where

  $$S_u = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{pmatrix}$$

  $$S_v = \begin{pmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

- The curvature

  $\kappa(a,b)$

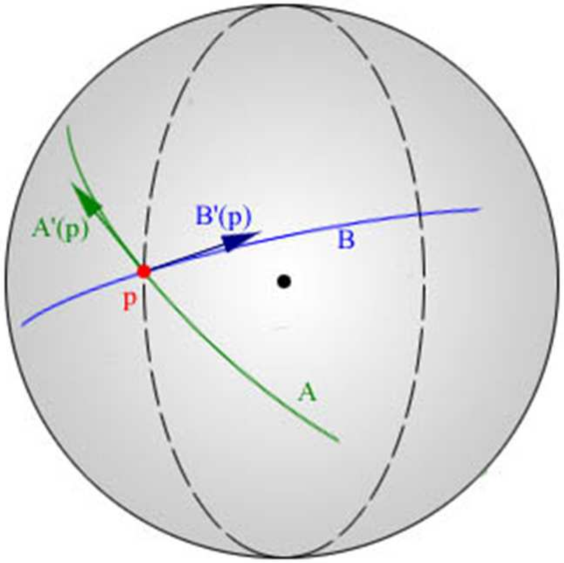


Image credit: www.peroxide.dk/.../tut10/pxdtut10.html

Curvature

- The curvature function is a quadratic function

$$\kappa(a,b) = la^2 + 2mab + nb^2$$
$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

- Where

$$l = S_{uu} \bullet N \quad m = S_{uv} \bullet N \quad n = S_{vv} \bullet N$$

Surface Curvature

$$\kappa(a,b) = la^2 + 2mab + nb^2$$

$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

- The quadric form has a maximum κ_1 and a minimum κ_2, which are the eigenvalues of the matrix.
- The corresponding eigenvectors are principle curvature directions.

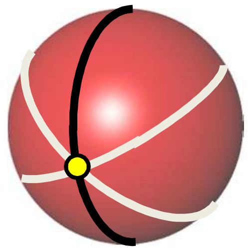
Discrete Principle Curvatures

- κ_1 and κ_2 satisfy $x^2 - 2Hx + K = 0$
- So $\kappa_{1,2} = H \pm \sqrt{H^2 - K}$

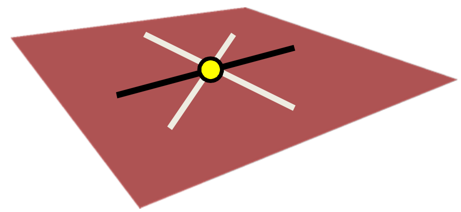
- Principle directions are found by finding the eigenvectors of the curvature tensor

Curvature Tensor

Isotropic



spherical

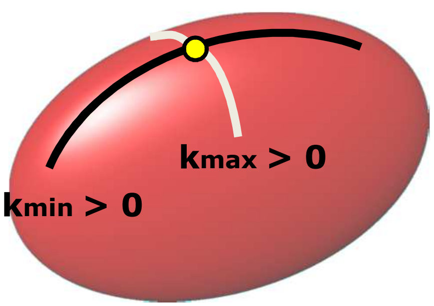


planar

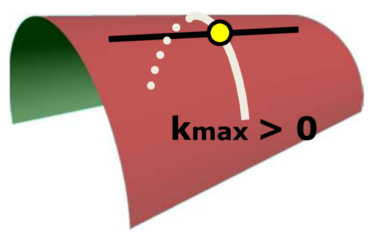
$\mathcal{T}(\mathbf{v})$

Anisotropic

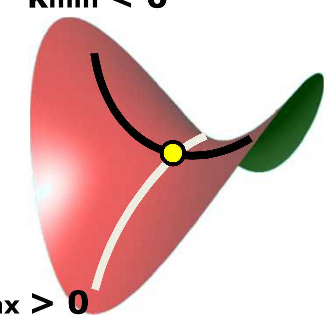
2 principal directions



elliptic



parabolic



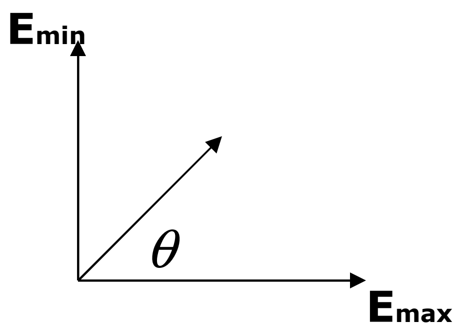
hyperbolic

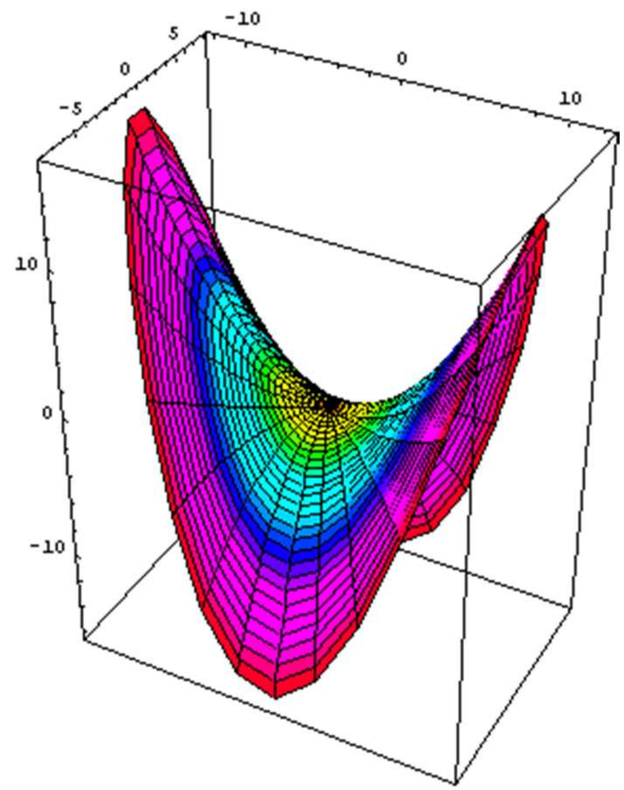
Image credit: Alliez et al.

Surface Curvature

- Some special numbers about the curvature tensor:
  - Normal curvature

$$\kappa_\theta = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$





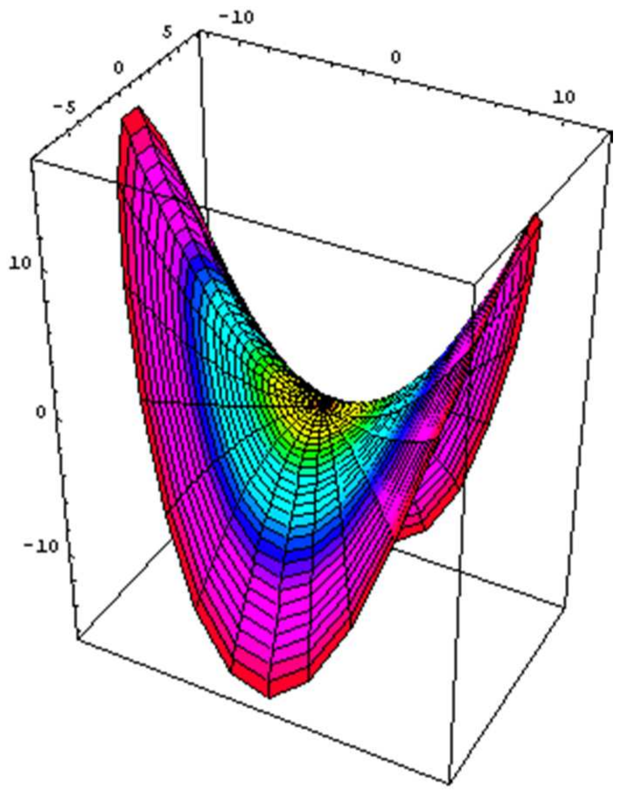
Surface Curvature

- Some special numbers about the curvature tensor:
  - Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

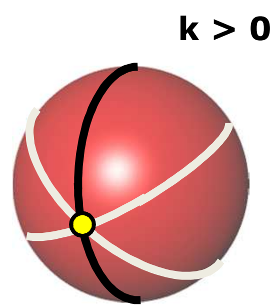
  - Gaussian curvature

$$K = \kappa_1 \kappa_2$$

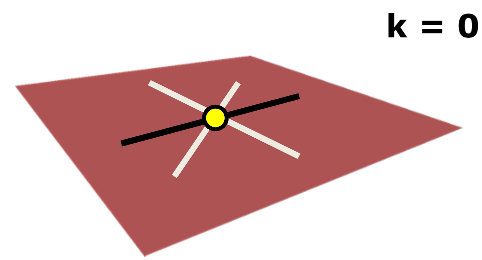


Curvature Tensor

Isotropic



spherical

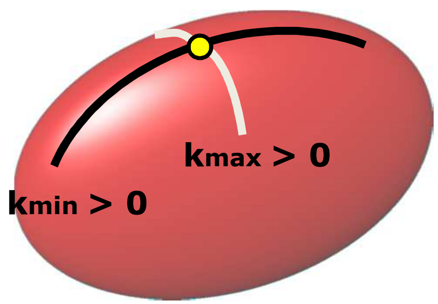


planar

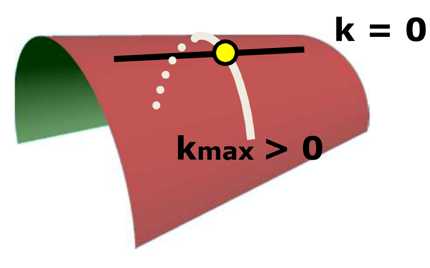
$\mathscr{T}(\mathbf{v})$

Anisotropic

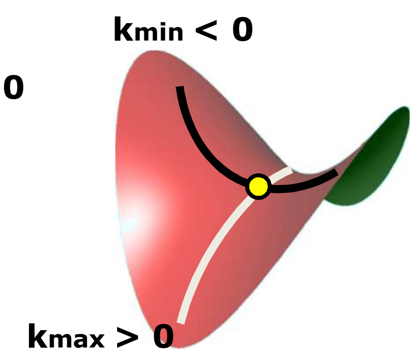
2 principal directions



elliptic



parabolic



hyperbolic

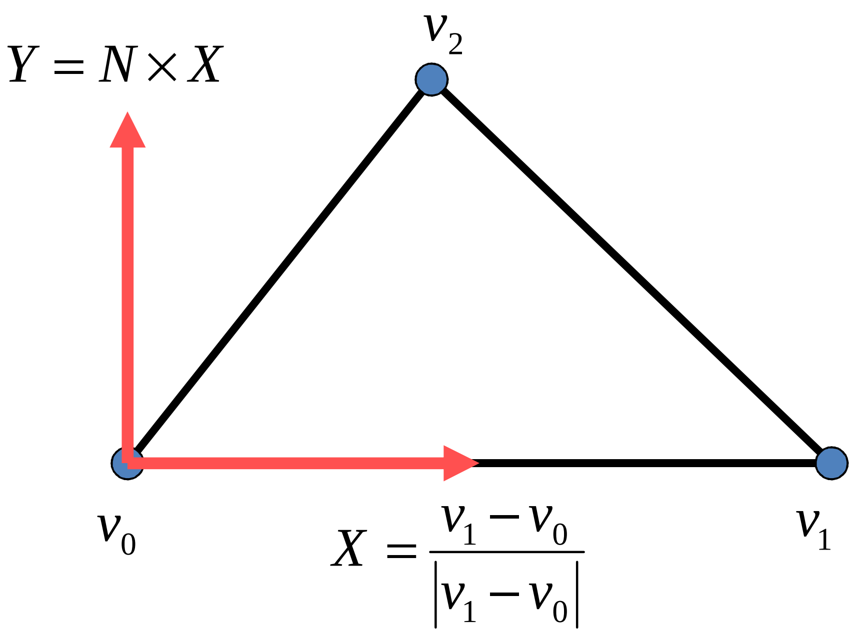
Image credit: Alliez et al.

How to Compute Curvature Tensor on a Mesh?

- Triangle:
  - Normal (well-defined, not continuous)
  - Curvature (zero inside a triangle)
- Vertex:
  - Normal (the average of the normal of the incident triangles)
  - Curvature

Local Frame

- Triangle



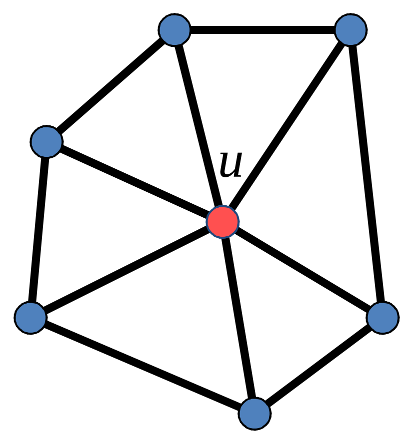
Local Frame

- Vertex
  - Find a 3D vector w

    $$X = N \times w$$

    $$Y = N \times X$$

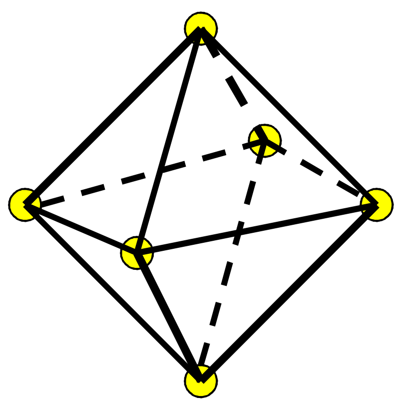
  - How do you find w?
    - Anyway is fine so long w is not co-linear with N

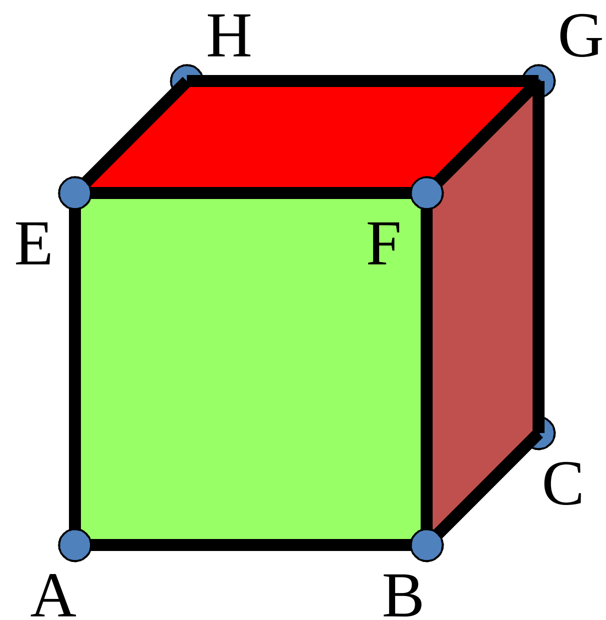


Discrete Gaussian Curvature

- Discrete Gaussian curvature for a vertex:

$$K(v) = 2\pi - \sum_{t \in \sigma(v)} \alpha(t_v)$$

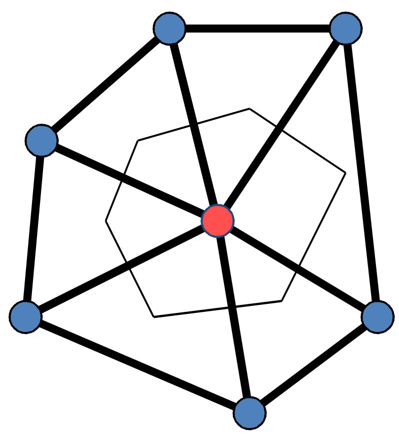




Discrete Gaussian Curvature

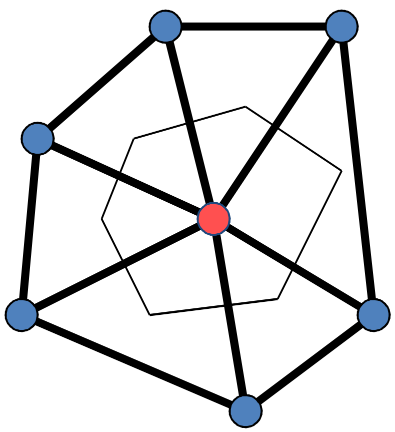
- But now the Gaussian curvature is not smooth.
- Treat the curvature at a vertex as a spatial average of its surrounding space

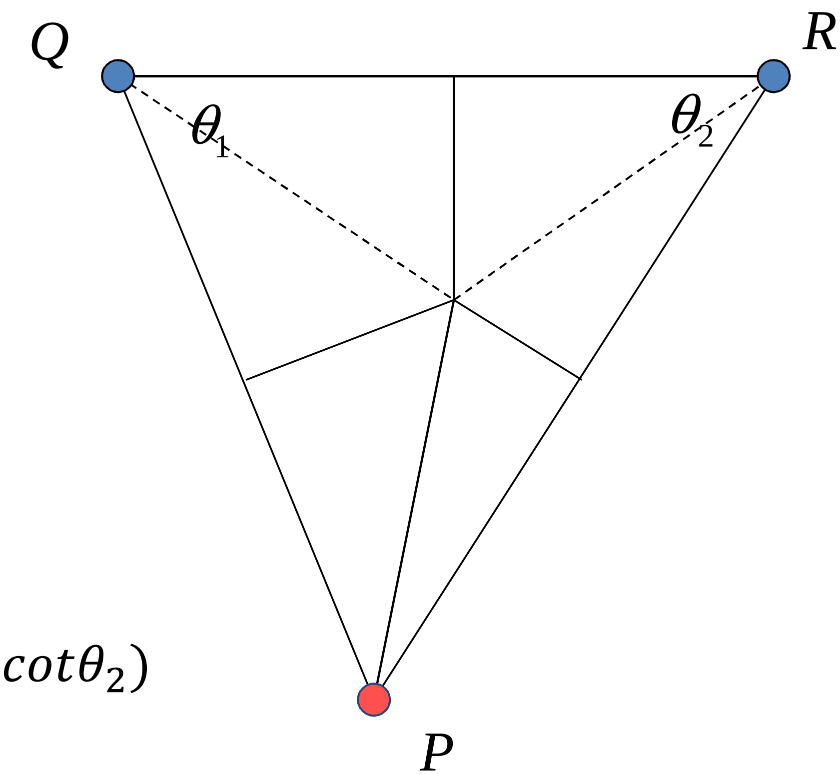
$$K(v) = \frac{2\pi - \sum_{t \in \sigma(v)} \alpha(t_v)}{Area(v)}$$



Voronoi Area Computation

- Non-obtuse (<π/2)

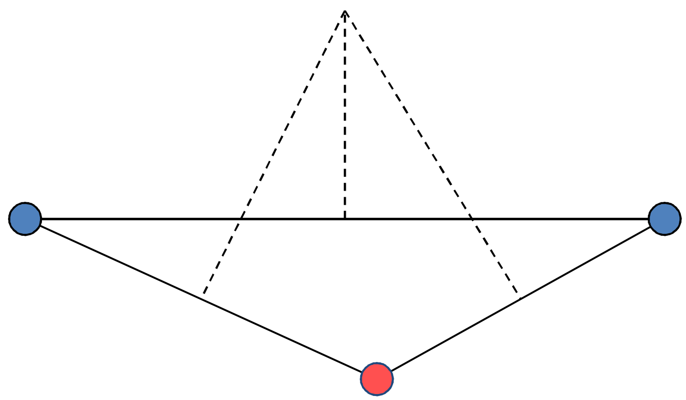




$$A_{Voronoi} = \frac{1}{8}(|PR|^2 cot\theta_1 + |PQ|^2 cot\theta_2)$$

Voronoi Area Computation

- Obtuse ($>\pi/2$)
  - Voronoi region is outside of the triangle



The algorithm for Voronoi Area Computation

```
𝒜_Mixed = 0
For each triangle T from the 1-ring neighborhood of x
    If T is non-obtuse,          // Voronoi safe
        // Add Voronoi formula (see Section 3.3)
        𝒜_Mixed += Voronoi region of x in T
    Else                         // Voronoi inappropriate
        // Add either area(T)/4 or area(T)/2
        If the angle of T at x is obtuse
            𝒜_Mixed += area(T)/2
        Else
            𝒜_Mixed += area(T)/4
```

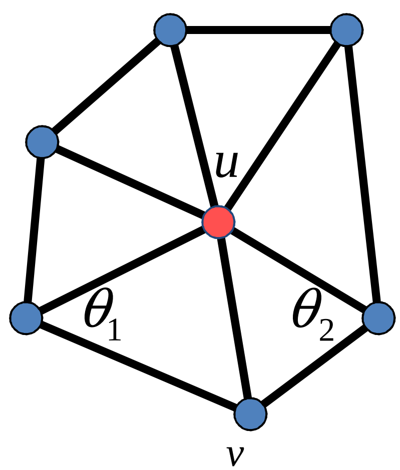
Discrete Mean Curvature

- Discrete mean curvature:

$$2H(u)N_u = \frac{\sum_{v \in \sigma(u)} \beta(u_v)}{Area(u)}$$

- where

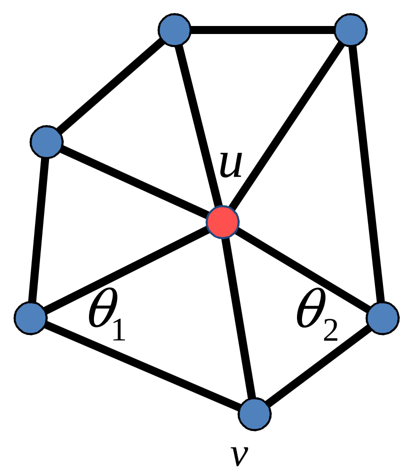
$$\beta(u_v) = \frac{(\cot\theta_1 + \cot\theta_2)}{2}(u - v)$$



Normal Curvature Along an Edge

- Discrete normal curvature for a vertex u along an edge (u, v) is:

$$\kappa_{u,v}^{\ \ N} = \frac{2(u-v) \bullet N_u}{\|u-v\|^2}$$



Mean Curvature and Normal Curvatures

- Discrete mean curvature at a vertex v is the weighted sum of the normal curvatures for edges incident to v:

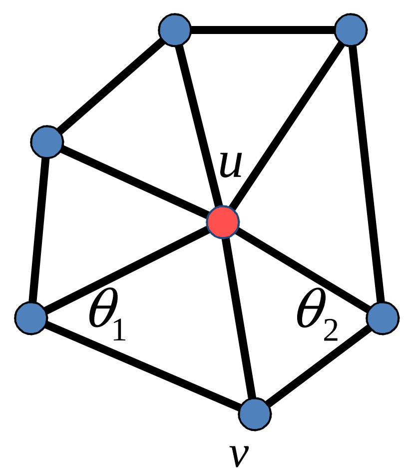
$$H(u) = \frac{1}{2}((2H(u)N_u) \bullet N_u) = \frac{\sum_{v \in \sigma(u)} \beta(u_v) \bullet N_u}{2Area(u)}$$

$$\beta(u_v) \bullet N_u = \frac{(\cot\theta_1 + \cot\theta_2)}{2}(u - v) \bullet N_u$$

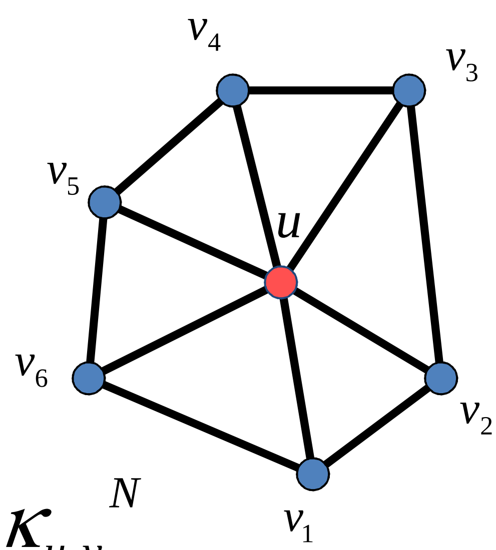
$$= \frac{(\cot\theta_1 + \cot\theta_2)}{2} \frac{\|u - v\|^2}{\|u - v\|^2}(u - v) \bullet N_u$$

$$= \frac{(\cot\theta_1 + \cot\theta_2)\|u - v\|^2}{4} \frac{2(u - v) \bullet N_u}{\|u - v\|^2}$$

$$= \frac{(\cot\theta_1 + \cot\theta_2)\|u - v\|^2}{4} \kappa_{u,v}{}^N$$



Discrete Curvature Tensor



- We know that:

$$\kappa(a,b) = la^2 + 2mab + nb^2$$

- Need to compute *l*, *m*, and *n*.
- For each edge (*u*, *v*), we also know $\kappa_{u,v}$
- Now we have a set of equations based on each edge:

$$\kappa_i = \kappa_{(a_i,b_i)} = la_i^2 + 2ma_ib_i + nb_i^2$$

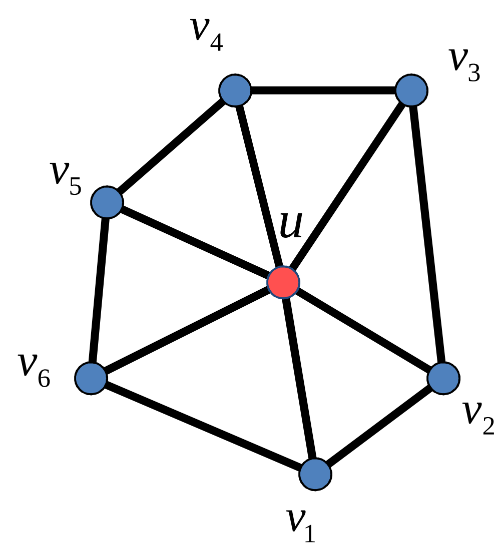
Discrete Curvature Tensor

- Solve the following system of linear equations:

$$\kappa_i = \kappa_{(a_i, b_i)} = l a_i^{\,2} + 2 m a_i b_i + n b_i^{\,2}$$

- or equivalently

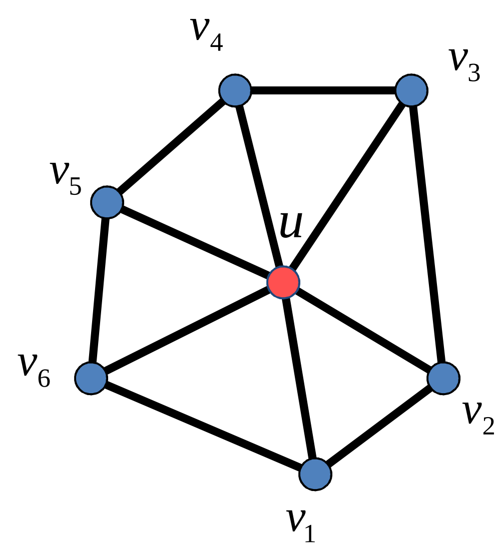
$$\begin{pmatrix} a_1^{\,2} & 2a_1b_1 & b_1^{\,2} \\ a_2^{\,2} & 2a_1b_1 & b_2^{\,2} \\ \vdots & \vdots & \vdots \\ a_n^{\,2} & 2a_nb_n & b_n^{\,2} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}$$



Discrete Curvature Tensor

- Solve the following system of linear equations:

$$\kappa_i = \kappa_{(a_i, b_i)} = l a_i^{2} + 2m a_i b_i + n b_i^{2}$$



- or equivalently

$$\begin{pmatrix} a_1^{2} & 2a_1 b_1 & b_1^{2} \\ a_2^{2} & 2a_1 b_1 & b_2^{2} \\ \vdots & \vdots & \vdots \\ a_n^{2} & 2a_n b_n & b_n^{2} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}$$

- The a’s and b’s are the 2D coordinates of tangent vectors.

Discrete Curvature Tensor

- How to solve it efficiently?

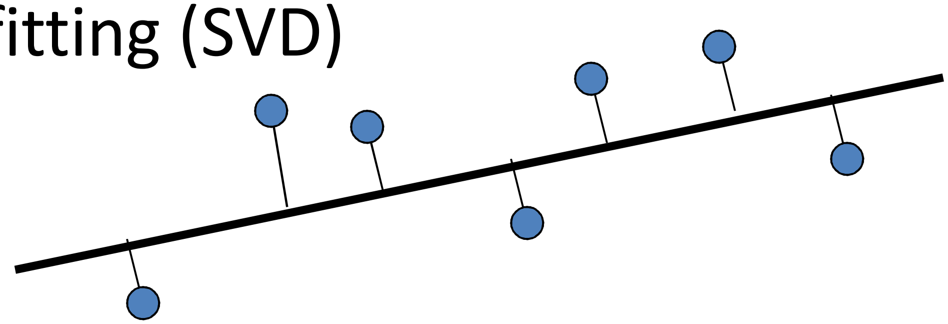
$$\begin{pmatrix} {a_1}^2 & 2a_1b_1 & {b_1}^2 \\ {a_2}^2 & 2a_1b_1 & {b_2}^2 \\ \vdots & \vdots & \vdots \\ {a_n}^2 & 2a_nb_n & {b_n}^2 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}$$

Discrete Curvature Tensor

- How to solve it efficiently?

$$\begin{pmatrix} a_1^{\ 2} & 2a_1b_1 & b_1^{\ 2} \\ a_2^{\ 2} & 2a_1b_1 & b_2^{\ 2} \\ \vdots & \vdots & \vdots \\ a_n^{\ 2} & 2a_nb_n & b_n^{\ 2} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}$$

- Least-square fitting (SVD)



Discrete Curvature Tensor

- How to solve it efficiently?

$$\begin{pmatrix} a_1^{\ 2} & 2a_1b_1 & b_1^{\ 2} \\ a_2^{\ 2} & 2a_1b_1 & b_2^{\ 2} \\ \vdots & \vdots & \vdots \\ a_n^{\ 2} & 2a_nb_n & b_n^{\ 2} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}$$

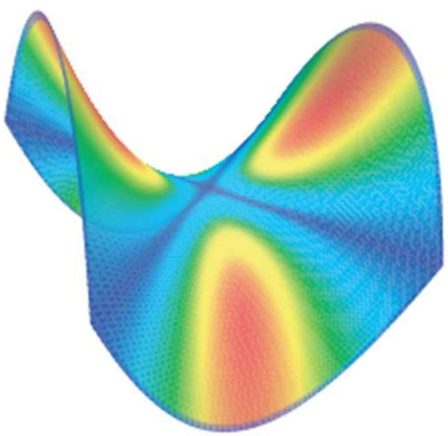
- Another way:

$$\begin{pmatrix} a_1^{\ 2} & 2a_1b_1 & b_1^{\ 2} \\ a_2^{\ 2} & 2a_1b_1 & b_2^{\ 2} \\ \vdots & \vdots & \vdots \\ a_n^{\ 2} & 2a_nb_n & b_n^{\ 2} \end{pmatrix}^T \begin{pmatrix} a_1^{\ 2} & 2a_1b_1 & b_1^{\ 2} \\ a_2^{\ 2} & 2a_1b_1 & b_2^{\ 2} \\ \vdots & \vdots & \vdots \\ a_n^{\ 2} & 2a_nb_n & b_n^{\ 2} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} a_1^{\ 2} & 2a_1b_1 & b_1^{\ 2} \\ a_2^{\ 2} & 2a_1b_1 & b_2^{\ 2} \\ \vdots & \vdots & \vdots \\ a_n^{\ 2} & 2a_nb_n & b_n^{\ 2} \end{pmatrix}^T \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix}$$

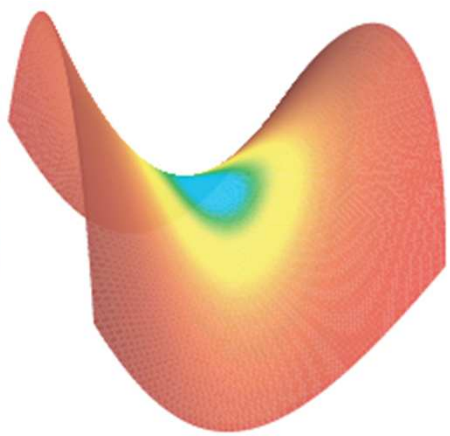
What Next?

- Put l,m,n into a 2x2 matrix and solve for eigenvalues (principle curvatures) and eigenvectors (principle directions)

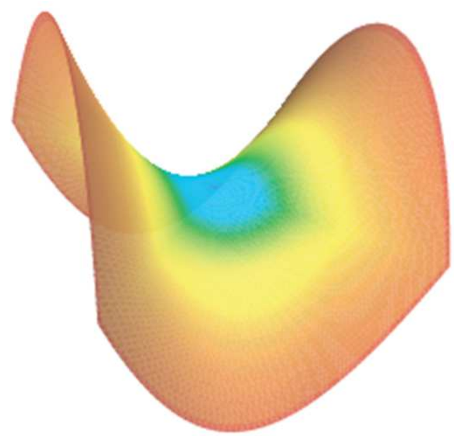
Examples



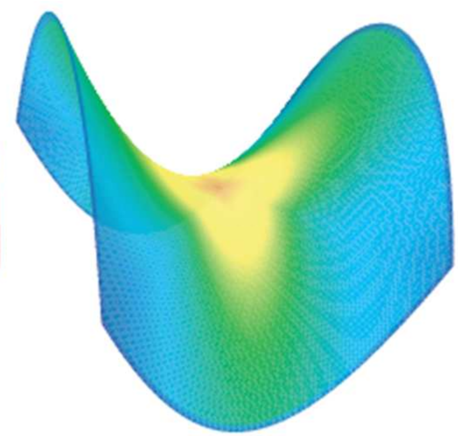
mean



Gaussian



minimum



maximum

Image credit: Mark Meyer et al.

Examples

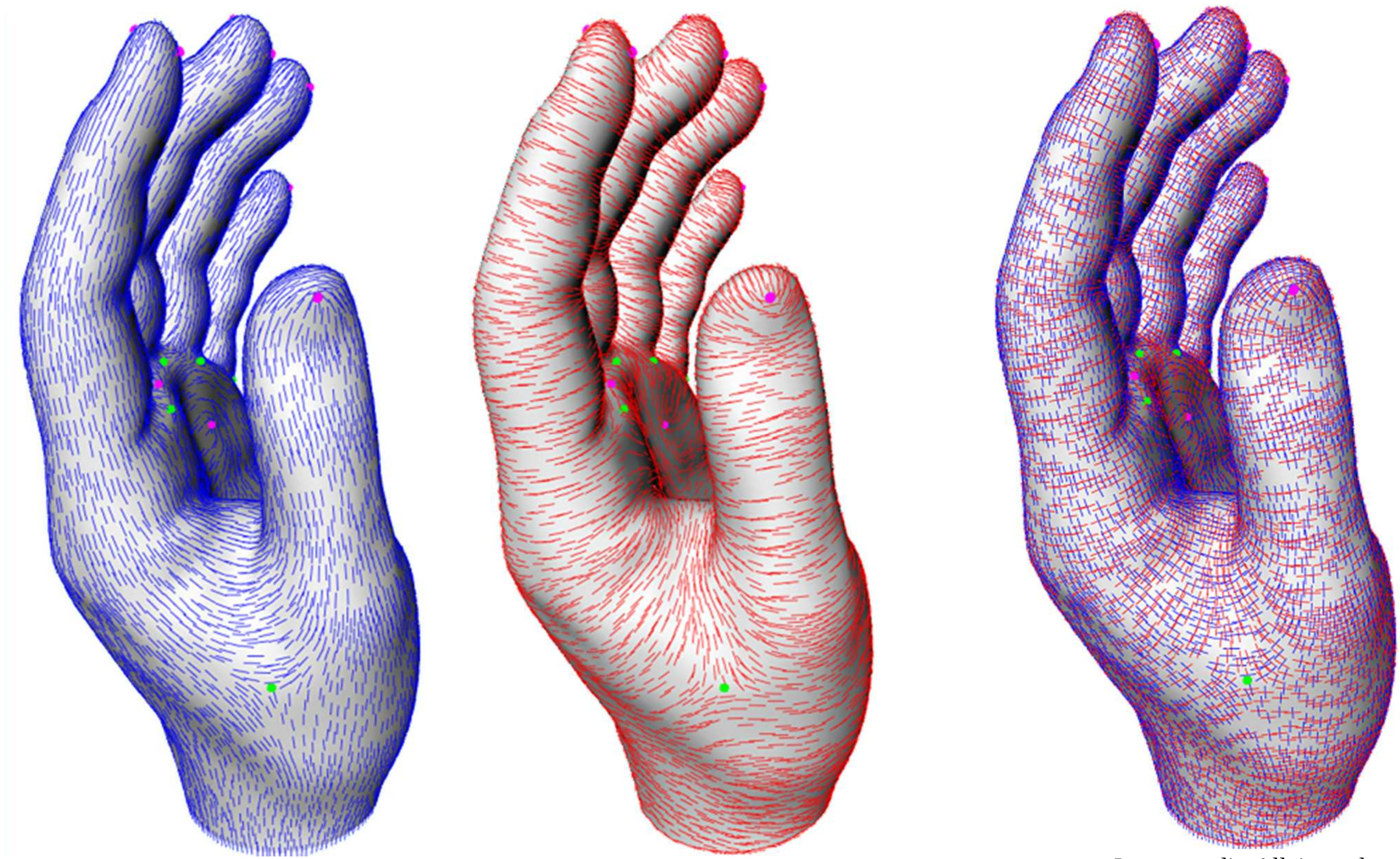
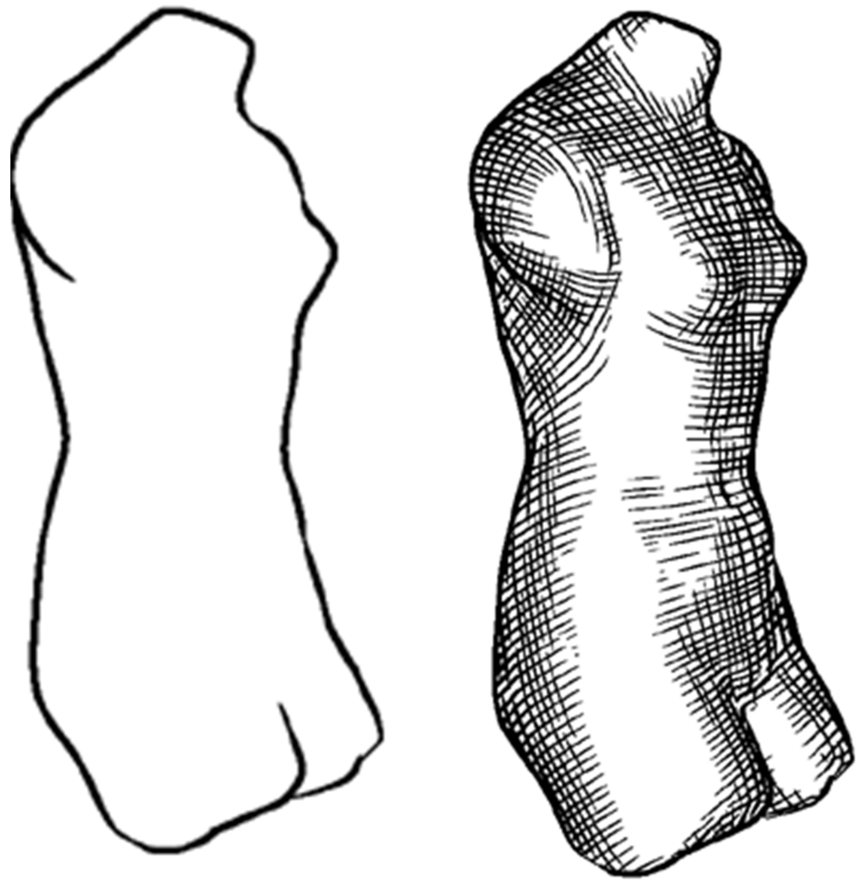


Image credit: Alleiz et al.

Hatch Drawing

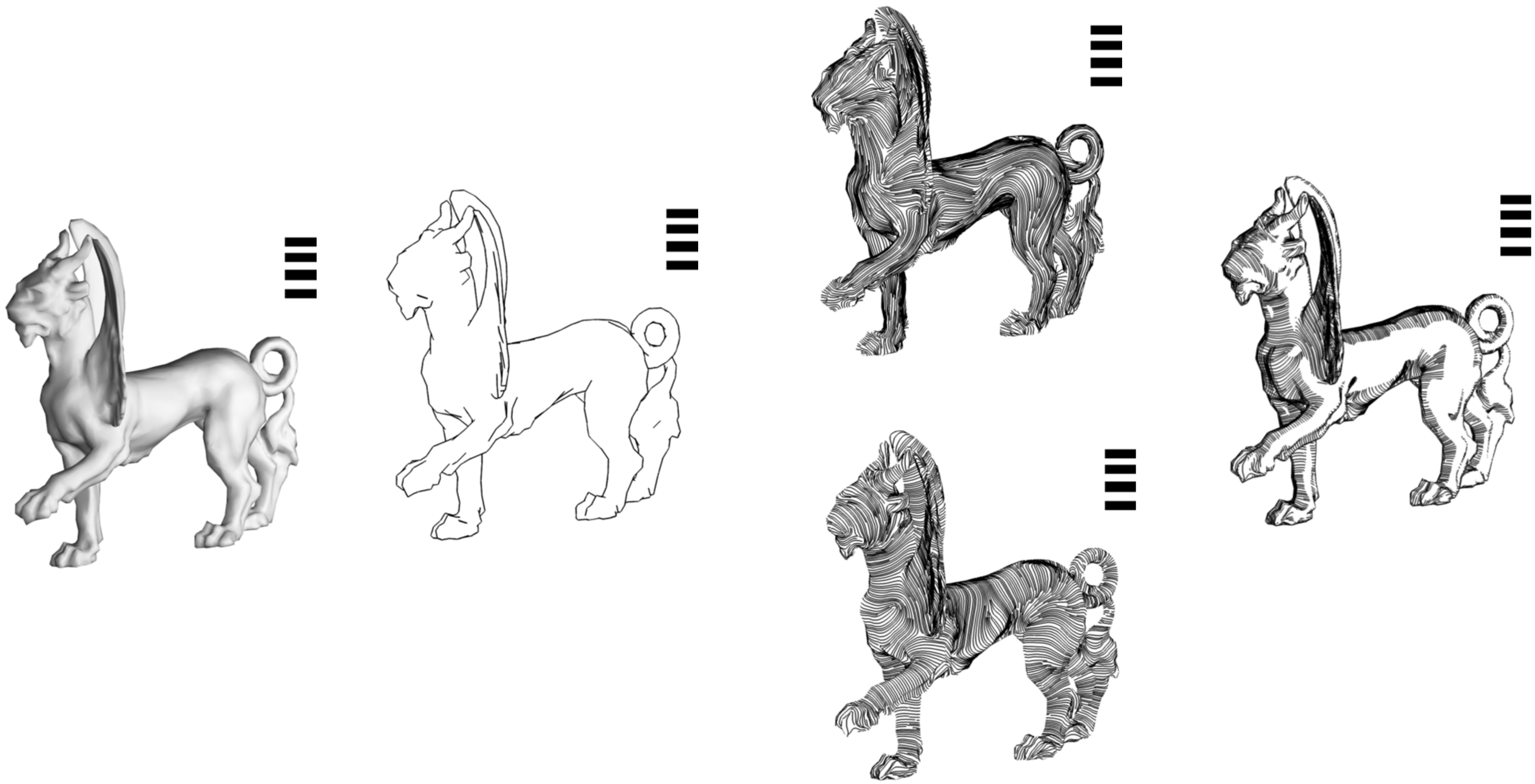
Praun et al.

A Comparison

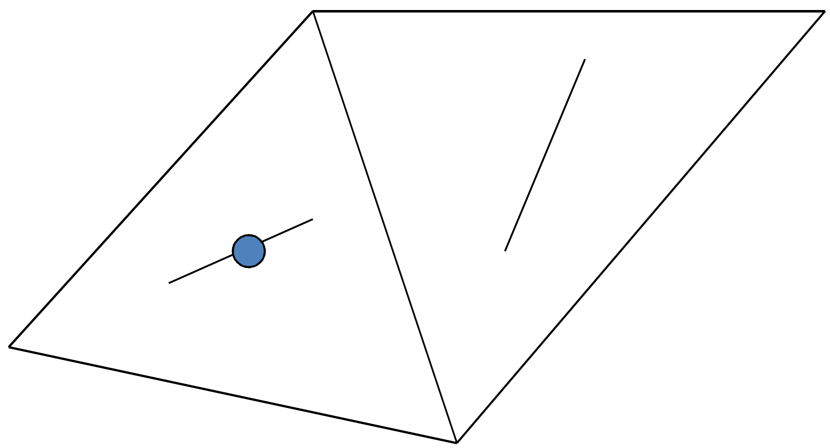


Hertzmann and Zorin

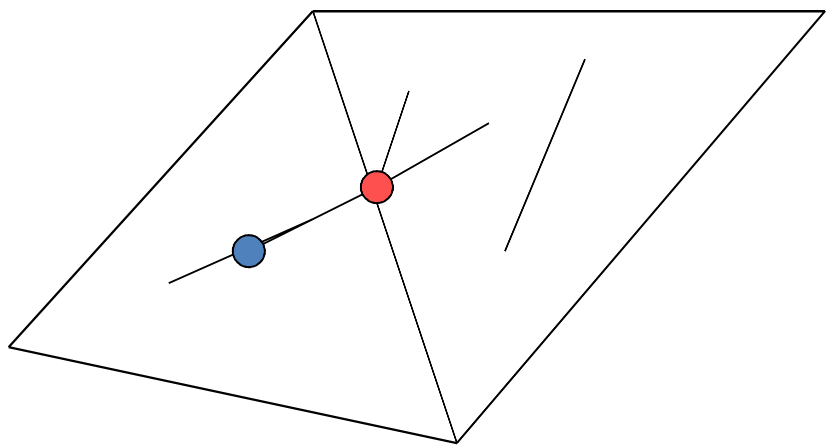
Rendering



Tracing Streamlines

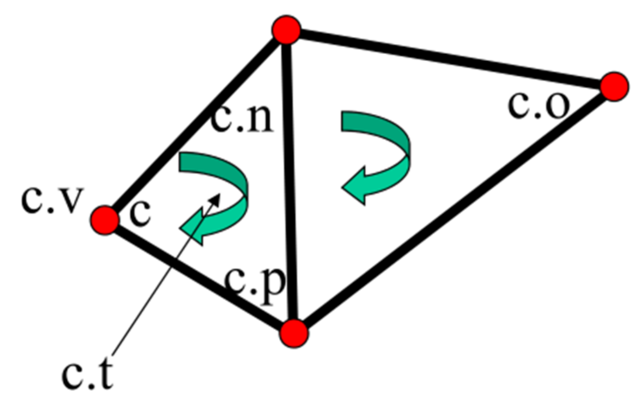


Tracing Streamlines



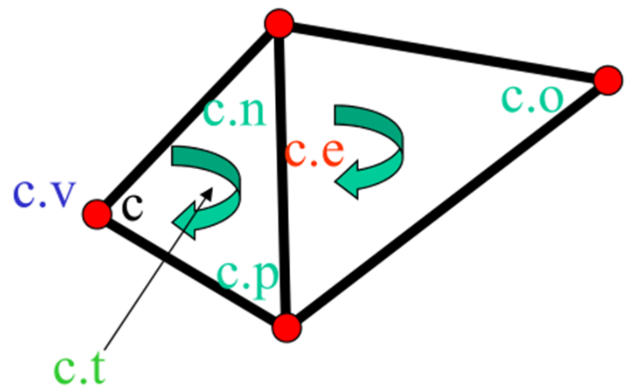
Corner Tables

- What is a corner?
- Why is it useful?
  - Angles
  - Other corner-related properties



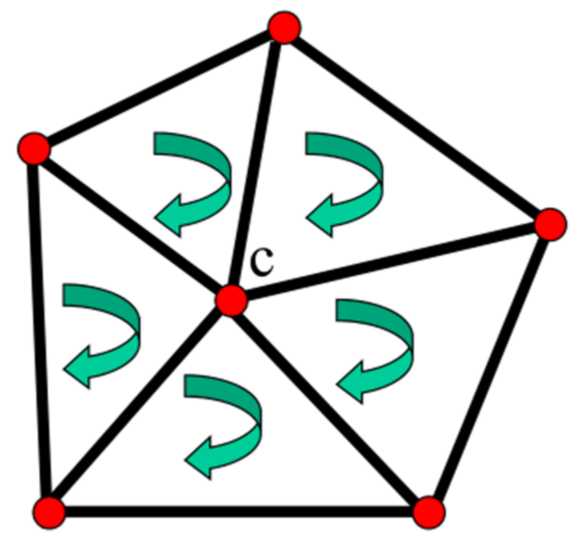
Corner Tables

- Operations
  - .p, .n, .o
  - .v
  - .e
  - .t
  - Can cascade:
  - What is c.o.t, c.o.p, c.o.n?
  - c.o.p=c.p.o?
  - c.o=NULL?



Corner Tables

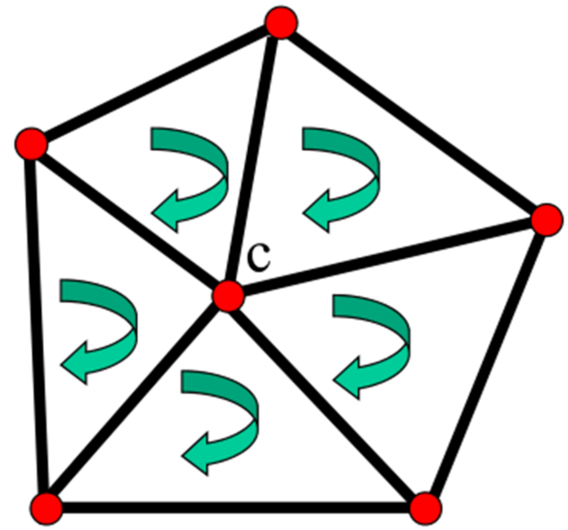
- Go around a vertex
- Where is c.p.o.n?
- Where is c.p.o.p?
- Where is (c.p.o.p).p.o.p?



Constructing Corner Tables

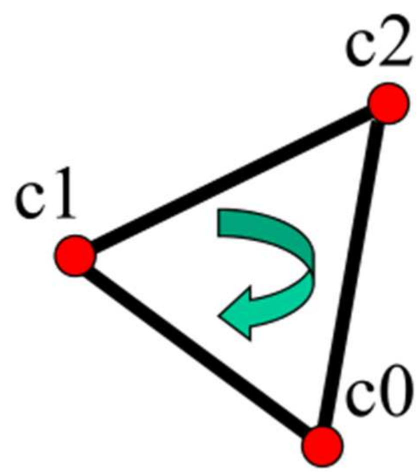
- Index
- .o, .p, .n, .v, .e, .t

```
class Corner{
public:
        unsigned char Edge_count;   //special variable for edges search 1/21/05
        int index;
        int v;              //the ID of the vertex of the corner
        int n;              //the next corner according to the orientation
        int p;              //the previous corner according to the orientation
        int t;              //the triangle the corner belongs to
        int ot;             //the index of its opposite triangle for traversal
        int o;              //the index of its opposite corner
        Edge *e;            //the opposite edge of the corner
        float angle;         //the angle of the corner
        float BeginAng, EndAng;  //for correct angle allocation of the corner around vertex v
        float r;
        bool orient;

        ///////////////////////////////////////////////////
        /* Optional variables */
        Edge *edge[2];      //two edges associated with this corner

        Corner()
        {
                e = NULL;
                edge[0] = edge[1] = NULL;
        }

        double get_Angle();
};
```



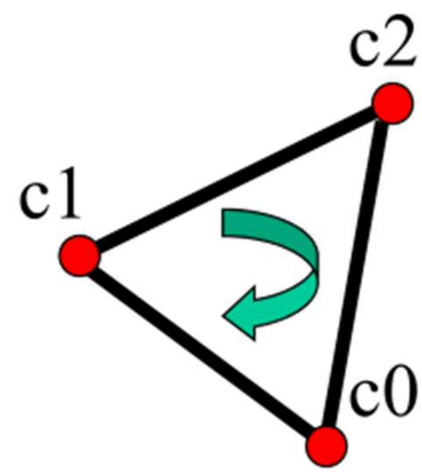
Constructing Corner Tables

- Read in all vertices and triangles
- Set num_corners = 3*num_tris
- For i=0 to num_tris
  - T=tlist[i]
  - T has three corners
    - c0=clist[3*i]
    - c1=clist[3*i+1]
    - c2=clist[3*i+2]
  - Such that ci.v=T.verts[i]



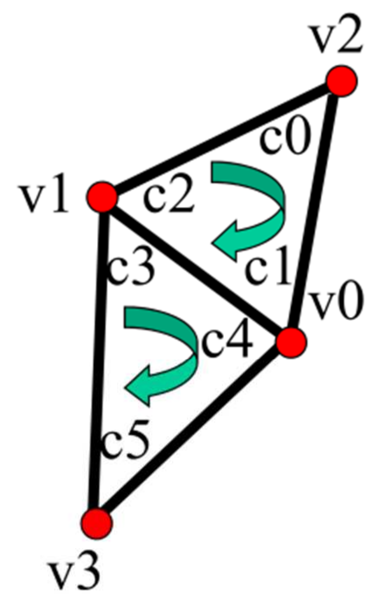
Constructing Corner Tables

- Construct the following table:
  - corner index, min(c.p.v.index, c.n.v.index), max(c.p.v.index, c.n.v.index)



Constructing This Table

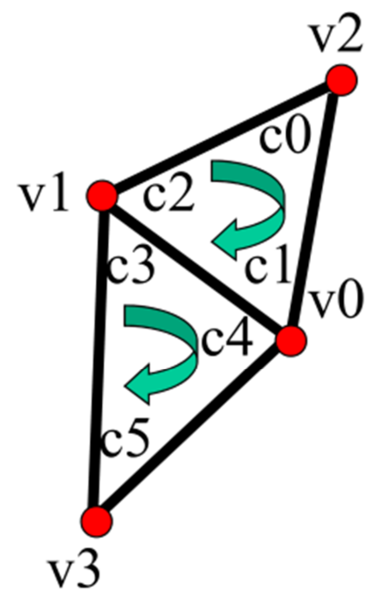
c.index	Min(c.p.v.index, c.n.v.index)	Max(c.p.v.index, c.n.v.index)	c.o.index
0	0	1	
1	1	2	
2	0	2	
3	0	3	
4	1	3	
5	0	1	



t0=(v2, v0, v1)
t1=(v1, v0, v3)

Sort According to Min/Max

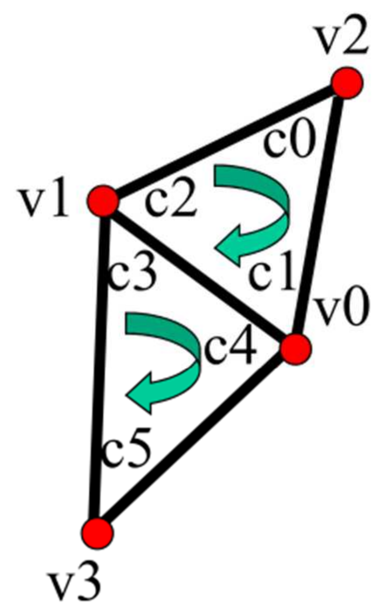
c.index	Min(c.p.v.index, c.n.v.index)	Max(c.p.v.index, c.n.v.index)	c.o.index
0	0	1	
5	0	1	
2	0	2	
3	0	3	
1	1	2	
4	1	3	



t0=(v2, v0, v1)
t1=(v1, v0, v3)

Look for Pairs and Set Up Links

c.index	Min(c.p.v.index, c.n.v.index)	Max(c.p.v.index, c.n.v.index)	c.o.index
0	0	1	5
5	0	1	0
2	0	2	NULL
3	0	3	NULL
1	1	2	NULL
4	1	3	NULL



t0=(v2, v0, v1)

t1=(v1, v0, v3)