# Vector Field Analysis

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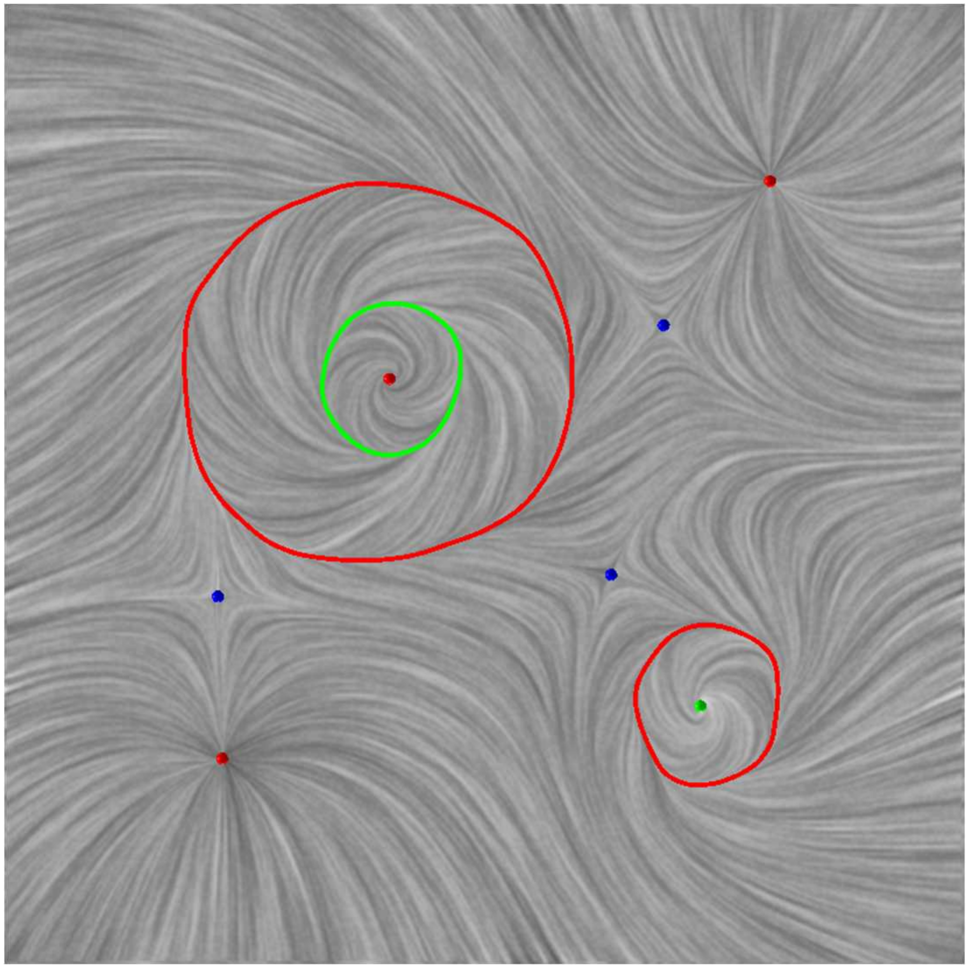
Similar to Morse theory, we analyze a vector field by classifying the behavior of its streamlines (i.e. integral lines)

# What Are We Looking For From Flow Data?

- For steady flow



# What Are We Looking For From Flow Data?

- For steady flow

Fixed points $V(x_0) = 0$

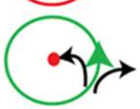
$\varphi(t, x_0) = x$ for all $t \in R$

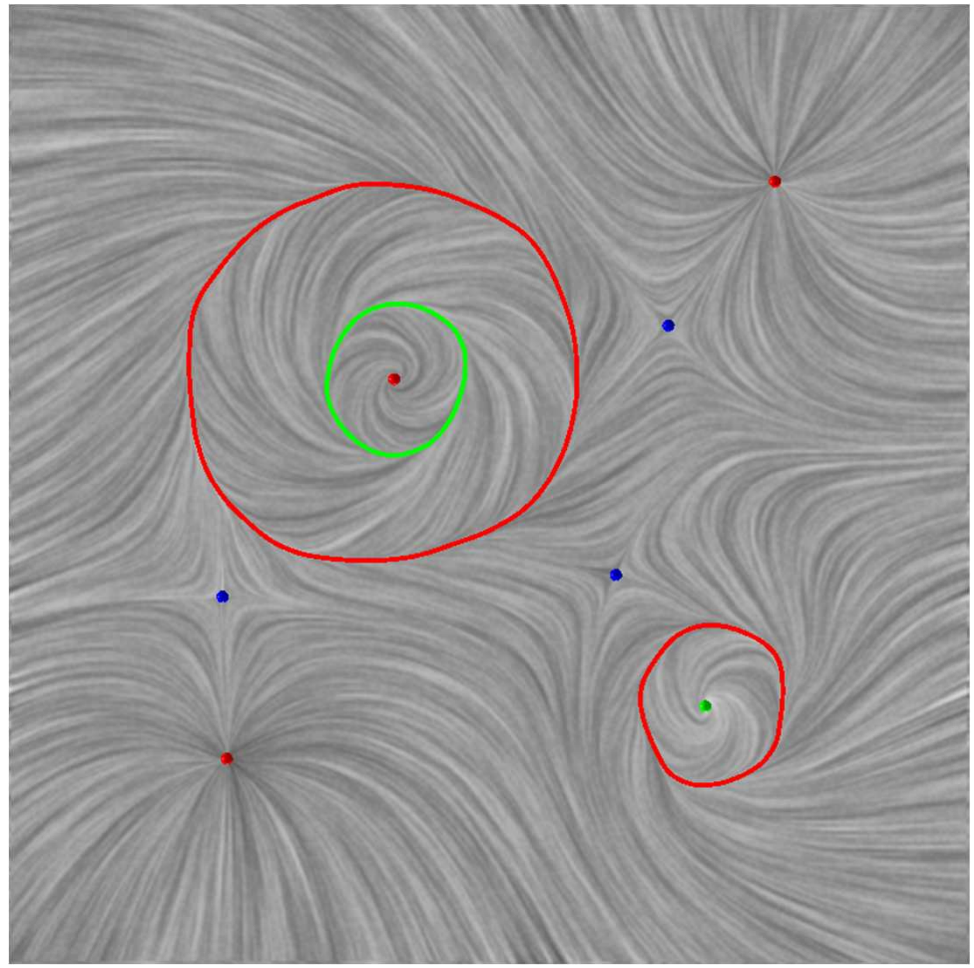
- Sink
- Source
- Saddle

Periodic orbits

$\exists T_0 > 0$ such that $\varphi(T_0, x) = x$

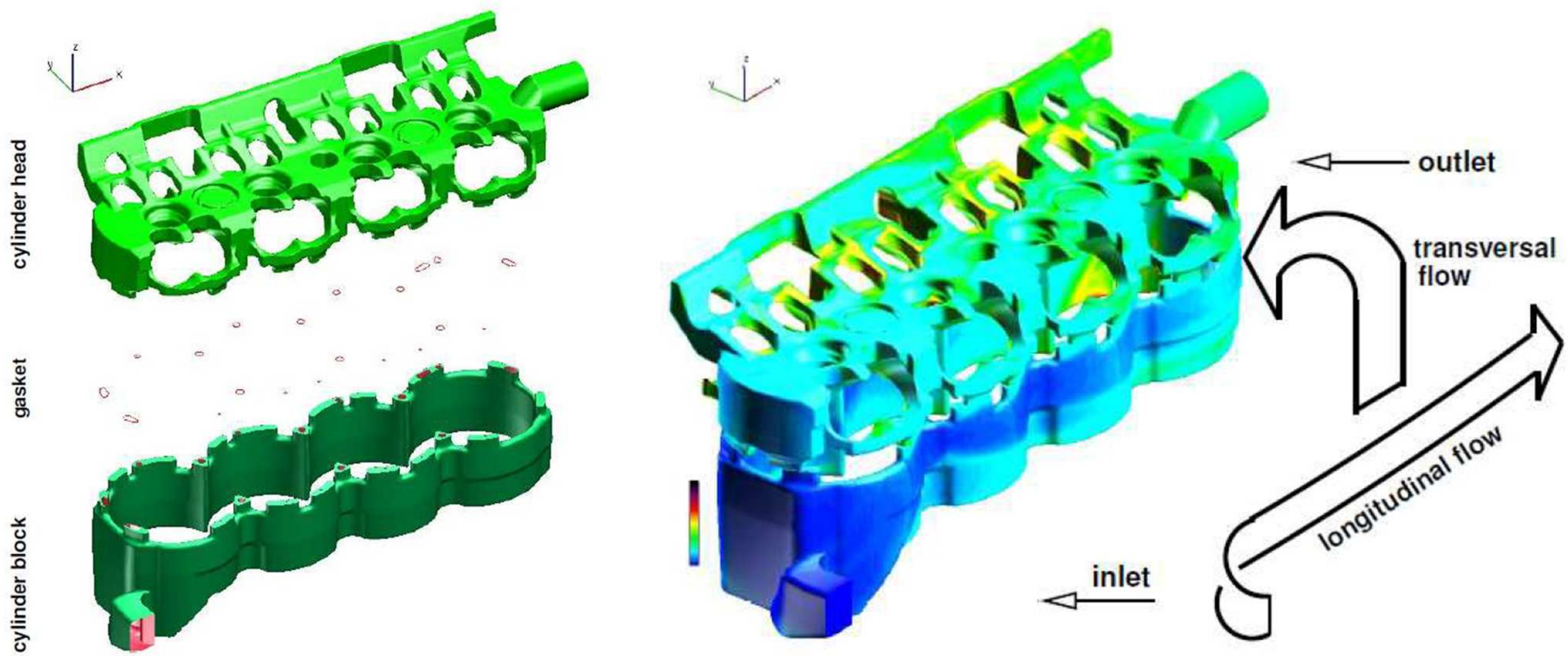
 Attracting

 Repelling



# Application in Automatic Design

- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary



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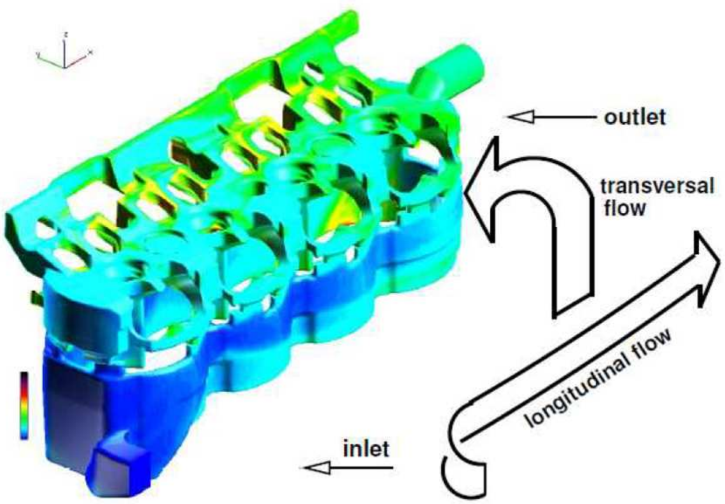
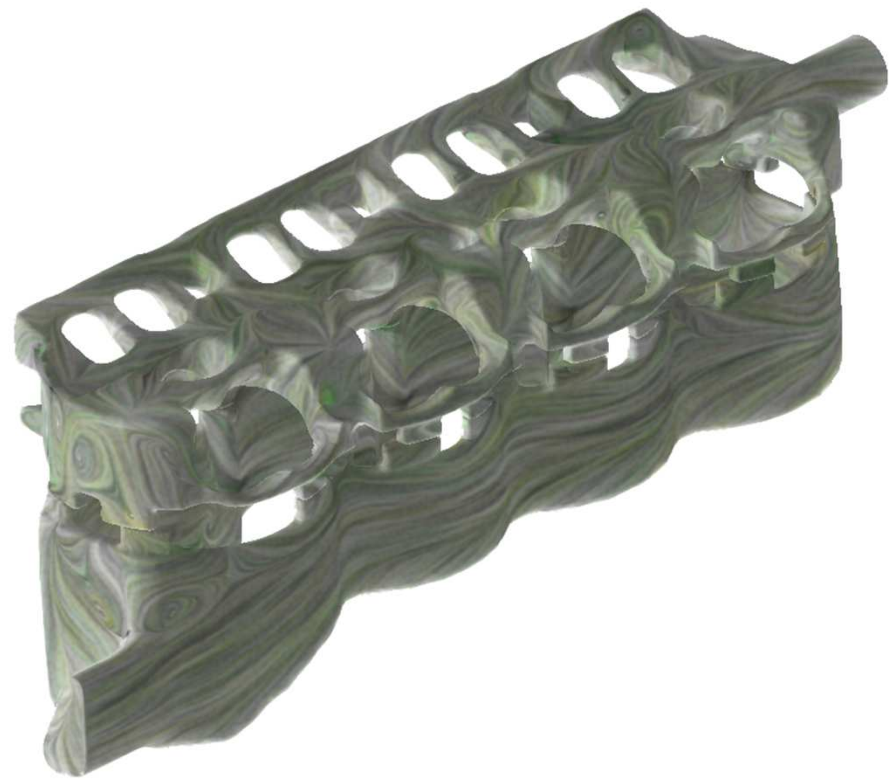
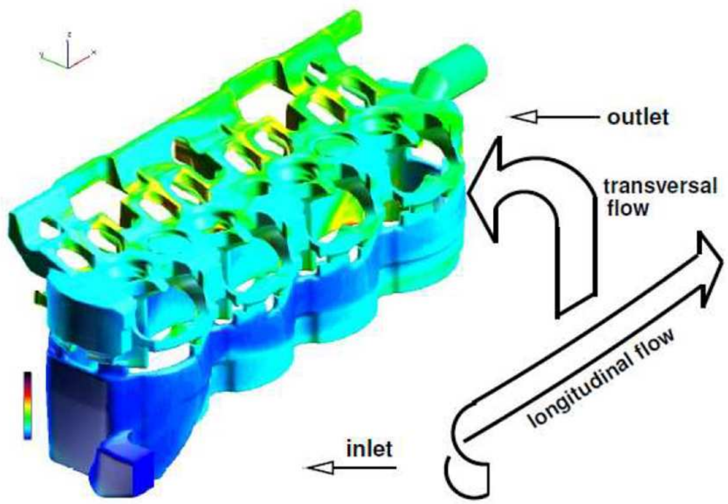


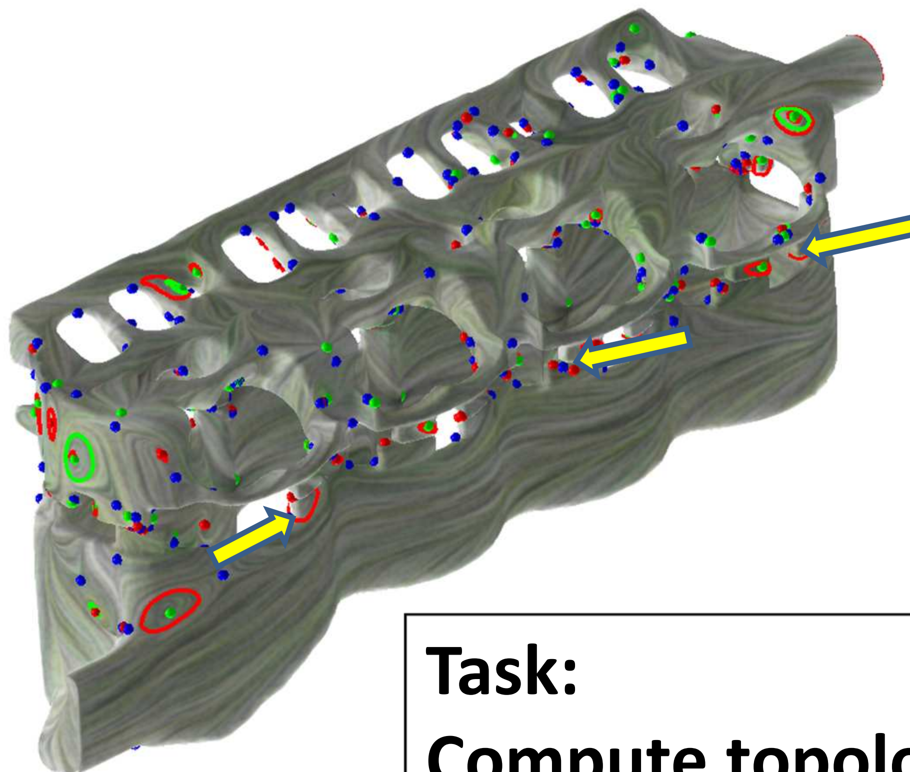



**Where are the critical dynamics of interests?**

# Vector Field Analysis

- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary

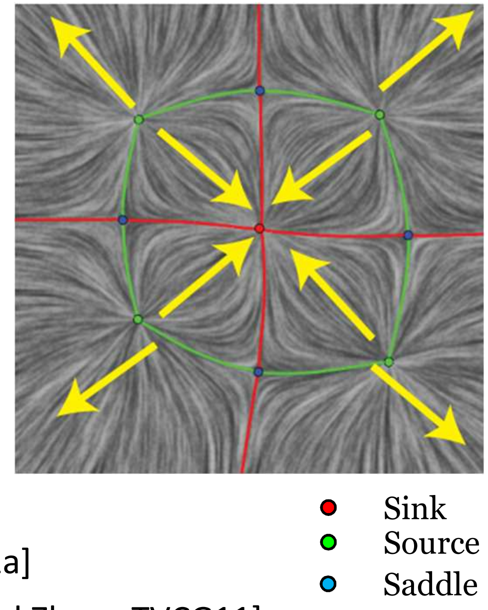


**Task:**
**Compute topology**

**These critical dynamics are parts of vector field topology!**

# Vector Field Topology





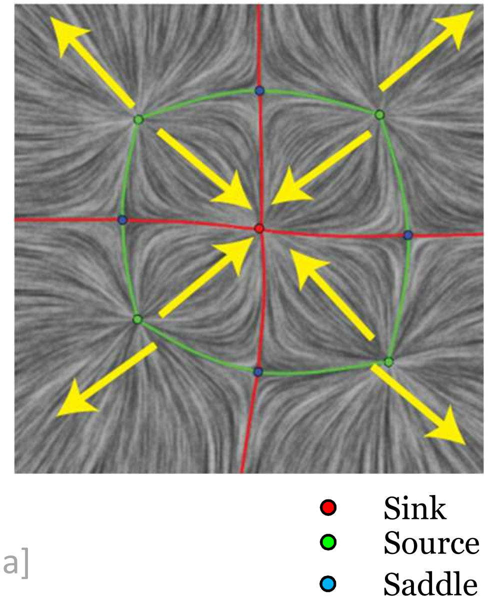
- Differential topology
  - Topological skeleton [Helman and Hesselink 1989; CGA91] [Scheuermann et al. Vis97, TVCG98][Tricoche et al. Vis01, VisSym01] [Theisel et al. CGF03][Polthier and Preuss 2003][Weinkauf et al VisSym04] [Weinkauf et al. Vis05]

- Discrete topology
  - Morse decomposition [Conley 78] [Chen et al. TVCG08, TVCG11a]
  - PC Morse decomposition [Szymczak EuroVis11] [Szymaczak and Zhang TVCG11]

- Combinatorial topology
  - Combinatorial vector field [Forman 98]
  - Combinatorial 2D vector field topology [Reininghaus et al. TopoInVis09, TVCG11]

# Vector Field Topology

- **Differential topology**
  - Topological skeleton [Helman and Hesselink 1989; CGA91] [Scheuermann et al. Vis97, TVCG98][Tricoche et al. Vis01, VisSym01] [Theisel et al. CGF03][Polthier and Preuss 2003][Weinkauf et al VisSym04] [Weinkauf et al. Vis05]

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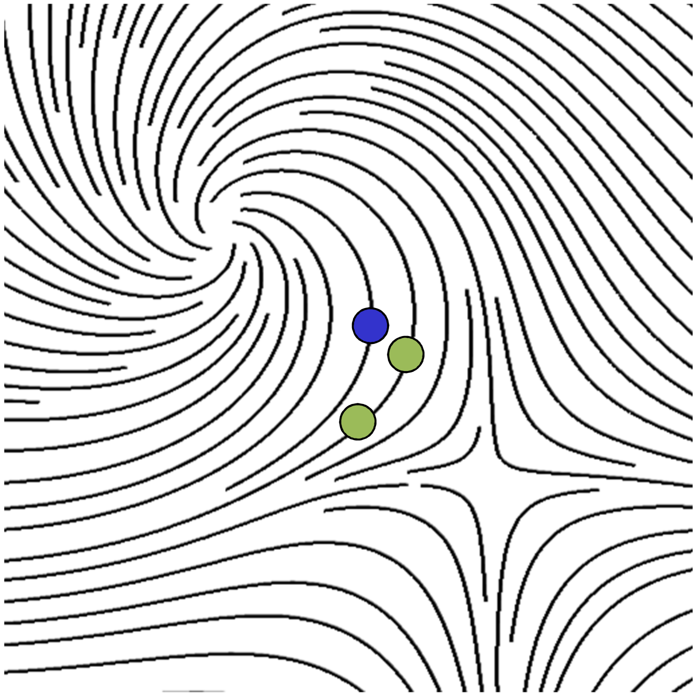
# Vector Fields (Recall)

- A **vector field**
  - is a continuous vector-valued function $V(x)$ on a manifold $X$
  - can be expressed as a system of ODE $\dot{x} = V(x)$
  - introduces a **flow** $\varphi : R \times X \to X$

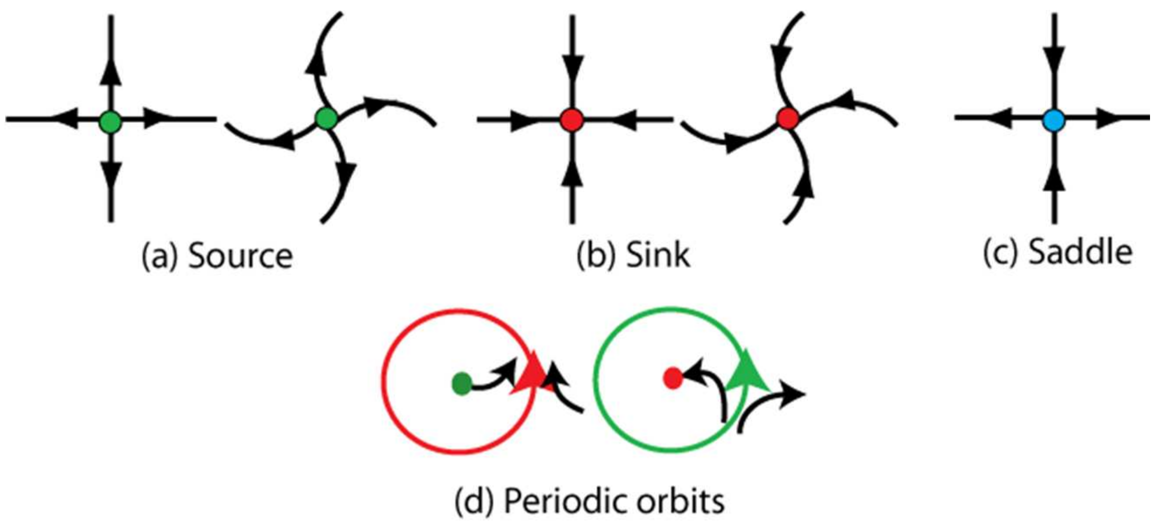
# Trajectories

- A **trajectory** of $x \in X$ is $\cup_{t \in R} \varphi(t, x)$
- Given an initial condition, there is a unique solution
  $\mathbf{x}(t) = \mathbf{x}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{x}(u))\, du$
  $\varphi(t_0) = \mathbf{x}_0$
- Under time-independent setting a trajectory is also called **streamline**



# Fixed Points and Periodic Orbits

- A point $x \in X$ is a **fixed point** if $\varphi(t, x) = x$ for all $t \in \boldsymbol{R}$
- $x$ is a periodic point if there exist a $T > 0$ such that $\varphi(T, x) = x$. The trajectory of a periodic point is called a **periodic orbit**.



# Limit Sets

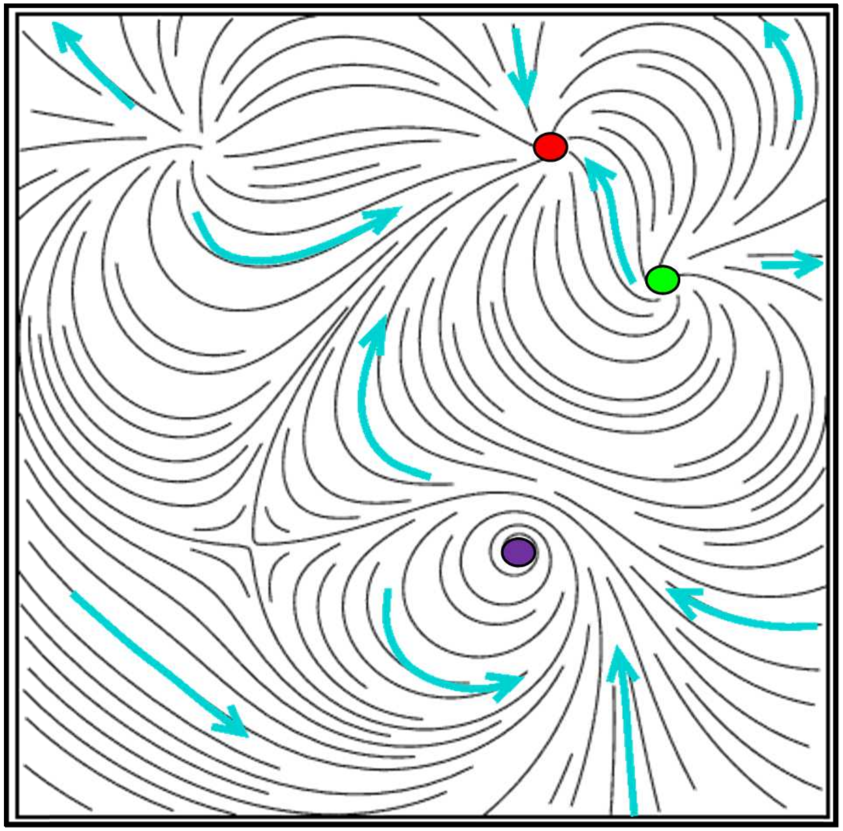
- Limit sets reveal the long-term behaviors of vector fields, correspond to flow recurrence
- The **limit sets** are:

$$\alpha(x) = \cap_{t<0}\, cl(\varphi((-\infty, t), x))$$

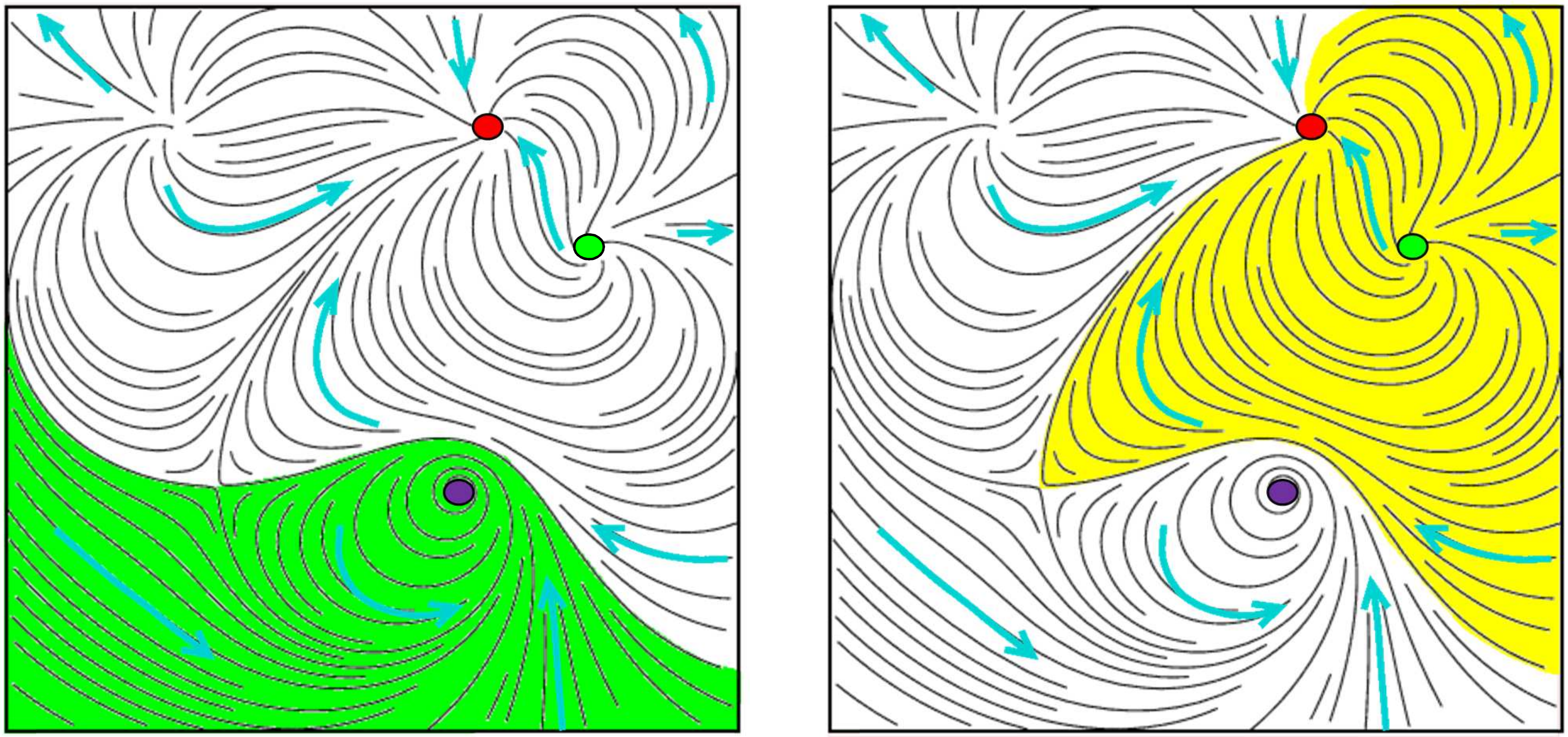
point (or curve) reached after ***backward*** integration by streamline seeded at x

$$\omega(x) = \cap_{t>0}\, cl(\varphi((t, \infty), x))$$

point (or curve) reached after ***forward*** integration by streamline seeded at x



# Repellor and Attractor Manifolds



# Invariant Sets

- An **invariant set** $S \subset X$ satisfies $\varphi(R,S)=S$
  - A trajectory is an invariant set
  - Fixed points and periodic orbits are invariant sets

# Compute Differential Topology

# Fixed Point Extraction

- Assume piecewise linear vector field. We adopt **cell-wise analysis**
  - We first locate the cells (e.g. triangles) that contain fixed points
    - solving linear / quadratic equation $\vec{v}(x, y) = \vec{0}$ to determine position of critical point in cell
  - Then, we classify the fixed points
    - **Jacobian** analysis

# Jacobian of 2D Vector Fields

- Assume a 2D vector field

$$ {d\mathbf{x}}/{dt} = V(\mathbf{x}) = \vec{\boldsymbol{f}}(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} ax + by + c \\ dx + ey + f \end{pmatrix} $$

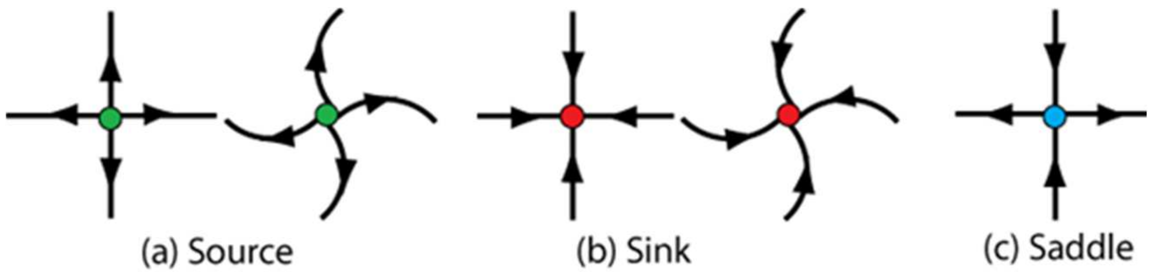
- Its **Jacobian** is

$$ \nabla V = \begin{bmatrix} \dfrac{\partial f_x}{\partial x} & \dfrac{\partial f_x}{\partial y} \\ \dfrac{\partial f_y}{\partial x} & \dfrac{\partial f_y}{\partial y} \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} $$

- **Divergence** is $a + e$
- **Curl** is $\mathrm{b} - \mathrm{d}$

Given a vector field defined on a discrete mesh, it is important to compute the coefficients a, b, c, d, e, f for later analysis.

# Fixed Point Classification




We specifically consider **first-order fixed points** -> Jacobian is **not degenerate**

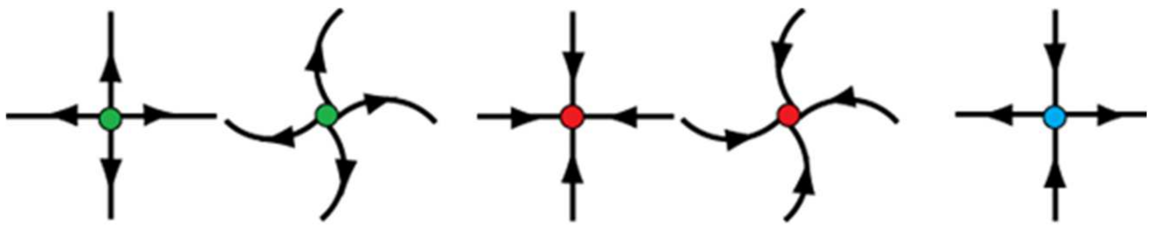
$\det(J) = ae - bd \neq 0$ -> Jacobian matrix is full rank

The eigenvalues of the Jacobian matrix $\lambda J = \lambda \boldsymbol{x}$ are

$$\lambda = Re_{1,2} + iIm_{1,2}$$

If both $Re_{1,2} > 0$, the fixed point repels flow locally.
If both $Re_{1,2} < 0$, the fixed point attracts flow locally.
If $Re_1 Re_2 < 0$, it does both and is a saddle

# Fixed Point Classification



(a) Source (b) Sink (c) Saddle

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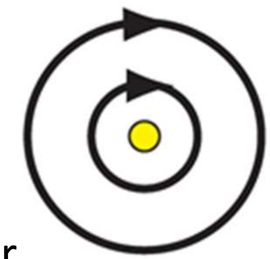
# Fixed Point Classification

Center

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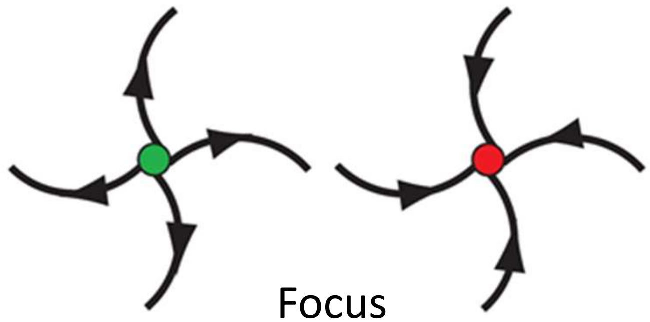
If both $Re_{1,2} = 0$ and $Im_{1,2} \neq 0$, the fixed point is **non-hyperbolic and unstable**.

# Fixed Point Classification



Focus

We specifically consider **first-order fixed points** -> Jacobian is **not degenerate**

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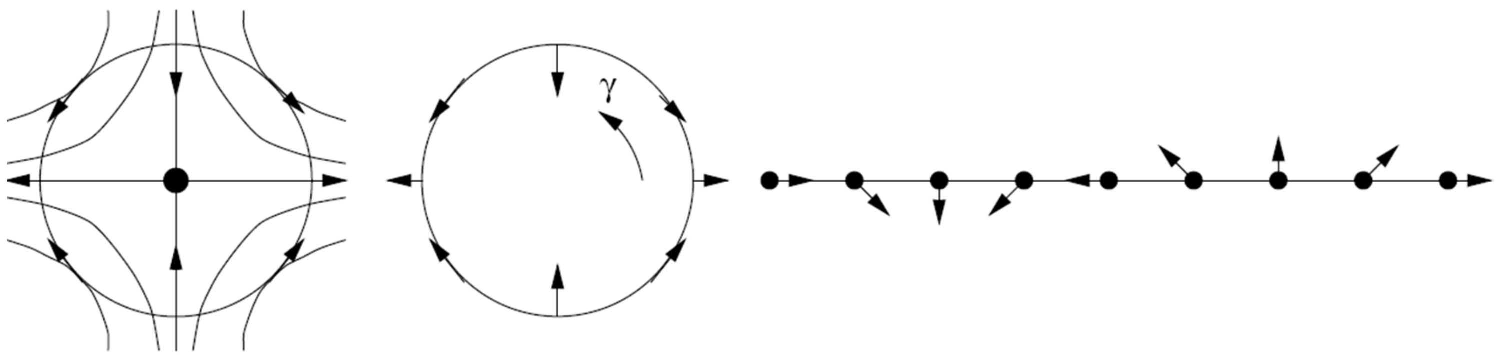
If either $Re_{1,2} \neq 0$, the fixed point is called **hyperbolic and stable**.
If both $Re_{1,2} = 0$ and $Im_{1,2} \neq 0$, the fixed point is **non-hyperbolic and unstable**.
A small perturbation will turn it into a stable (hyperbolic) fixed point

An alternative way to locate fixed points based on Poincaré Index

# Poincaré Index

- **Poincarè index** I($\Gamma, V$) of a simple closed curve $\Gamma$ in the plane relative to a continuous vector field is the number of the positive field rotations while traveling along $\Gamma$ in positive direction.

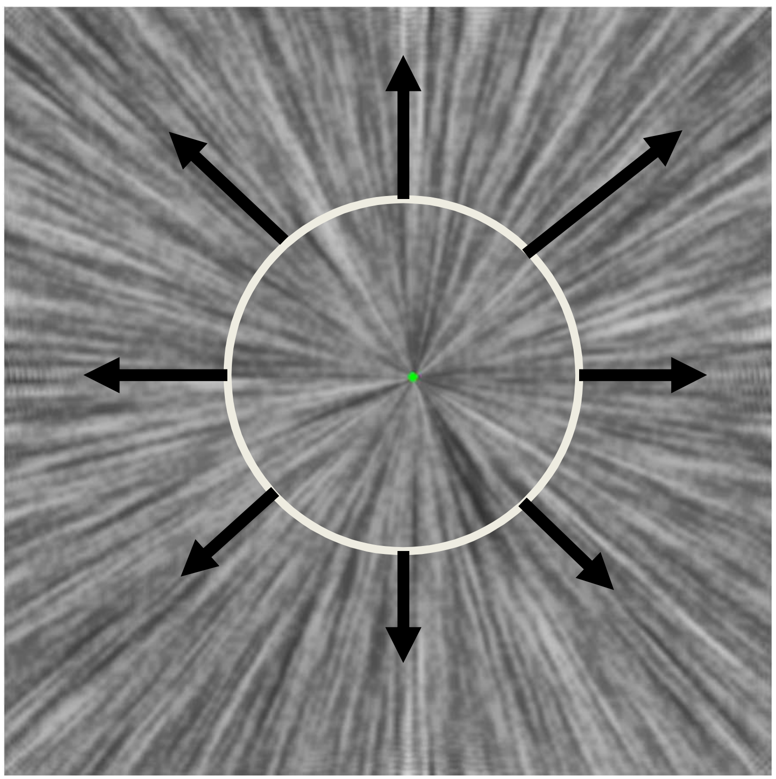


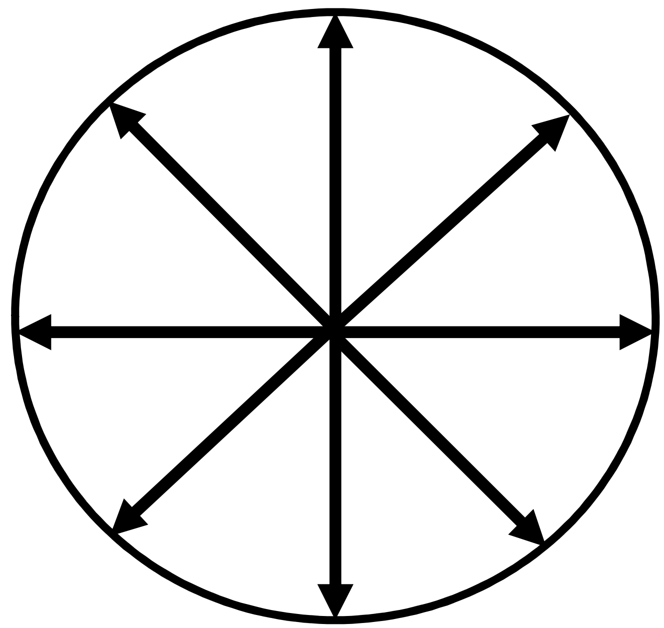

[Tricoche Thesis2002]

- By continuity, always an integer
- The index of a closed curve around multiple fixed points will be the sum of the indices of the fixed points

# Poincaré Index

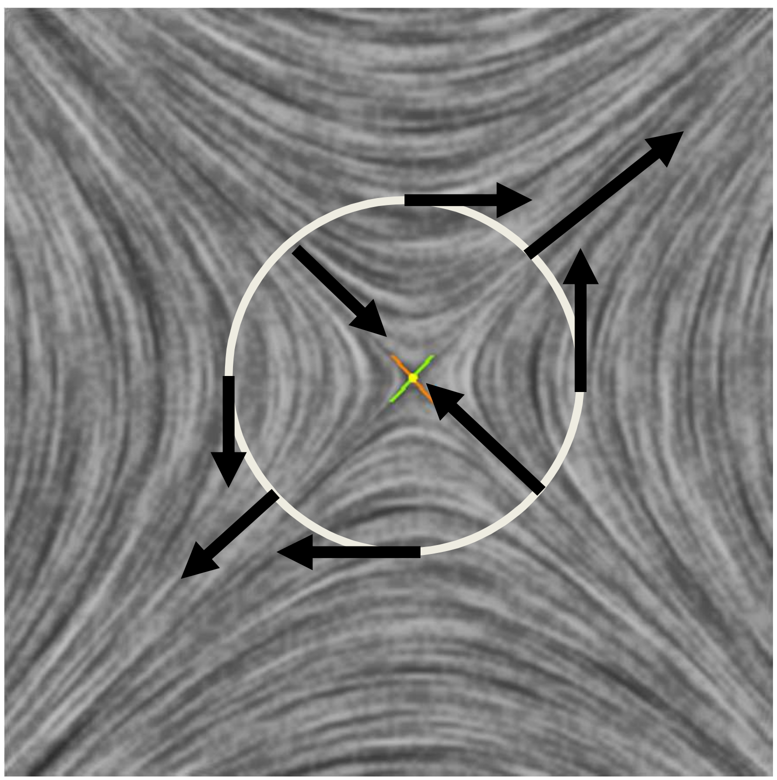
- Poincaré index: sources

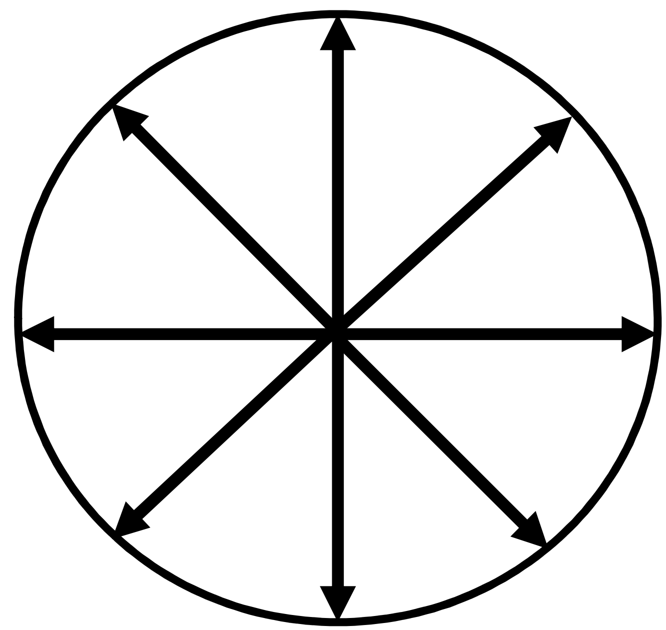




# Poincaré Index

- Poincaré index: saddles





# Important Poincaré Indices

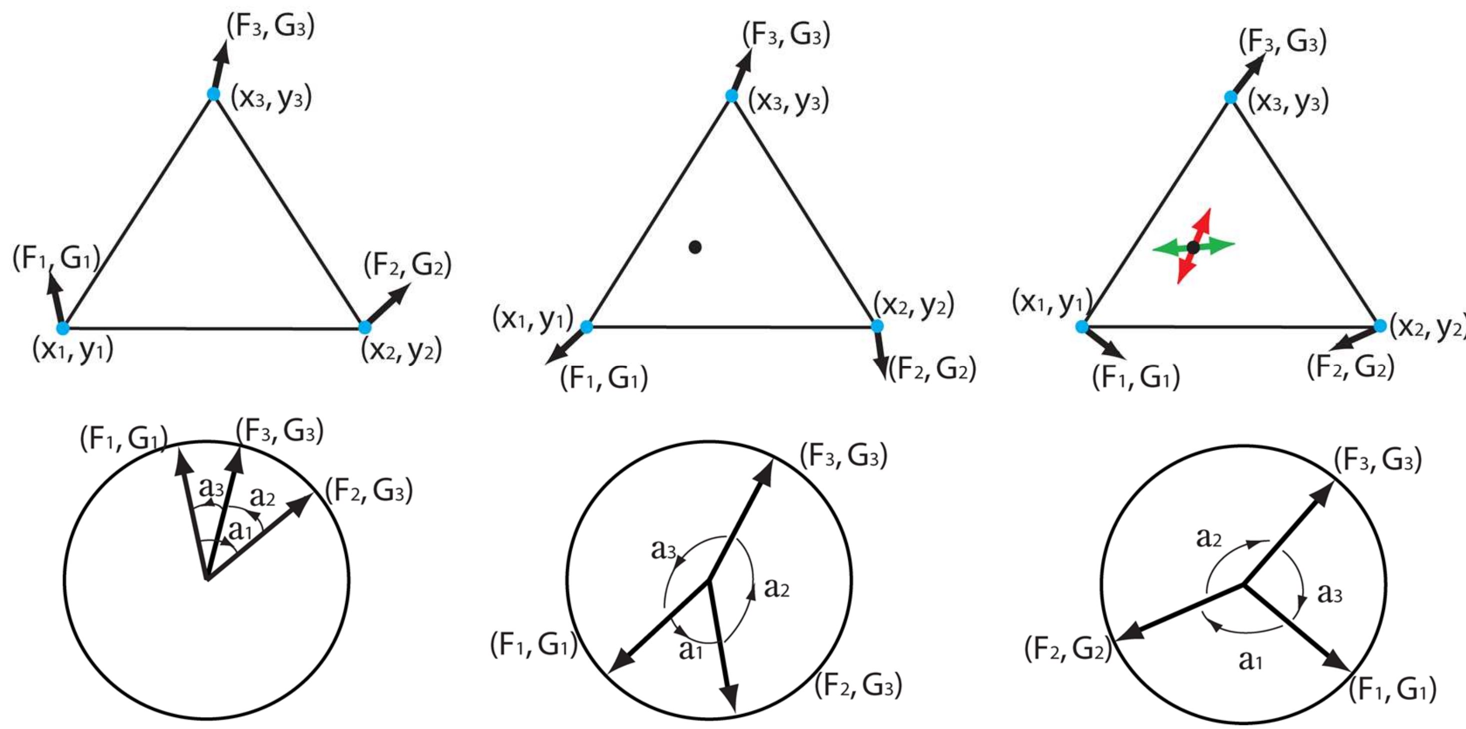
- Consider an **isolated** fixed point $x_0$, there is a neighborhood $N$ enclosing $x_0$ such that there are no other fixed points in $N$ or on the boundary curve $\partial N$
  - if $\mathbf{I}(\partial N, V) = 1$, $x_0$ is either a source or a sink;
  - if $\mathbf{I}(\partial N, V) = -1$, $x_0$ is a saddle.

- The Poincarè index of a fixed point free region is $0$

# Locate Fixed Point Based on Poincaré Index

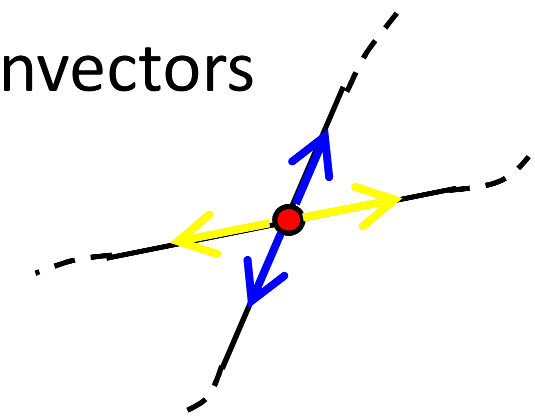
For each triangle, compute the total rotation of the vector field along its boundary. This is based on the assumption that the triangle mesh is oriented.

The triangles that contain fixed points (i.e. non-trivial index) are marked singular, where we solve the linear system to compute the location of the fixed point



# Fixed Point Extraction

- Assume piecewise linear vector field. We adopt **cell-wise analysis**
  - We first locate the cells (e.g. triangles) that contain fixed points
  - Then, we classify the fixed points
    - **Jacobian** analysis
  - If type is saddle, compute eigenvectors



# Summary of Fixed Point Extraction

- Assume piecewise linear vector field. We adopt **cell-wise analysis**
  - We first locate the cells (e.g. triangles) that contain fixed points using Poincaré index
  - Then, we classify the fixed points
    - **Jacobian** analysis
    - solving linear system to determine the coefficients (a,b,c,d,e,f)
    - The position of the fixed point is

$$\begin{cases} x = \dfrac{bf - ce}{ae - bd} \\ y = \dfrac{cd - af}{ae - bd} \end{cases}$$

  - If type is saddle, compute eigenvectors of the Jacobian matrix

# One More Thing for Poincaré Indices

- Consider an **isolated** fixed point $x_0$, there is a neighborhood $N$ enclosing $x_0$ such that there are no other fixed points in $N$ or on the boundary curve $\partial N$
  - if $\mathbf{I}(\partial N, V) = 1$, $x_0$ is either a source or a sink;
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- The Poincarè index of a fixed point free region is $0$

- There is a combinatorial theory that shows

$$I = 1 + \frac{e - h}{2}$$

# Sectors & Separatrices

- In the vicinity of a fixed point, there are various *sectors* or regions of different flow type:
  - *hyperbolic*: paths do not ever reach fixed point.
  - *parabolic*: one end of all paths is at fixed point.
  - *elliptic*: all paths begin & end at fixed point.
- A *separatrix* is the bounding curve (or surface) which separates these regions

# Sectors & Separatrices

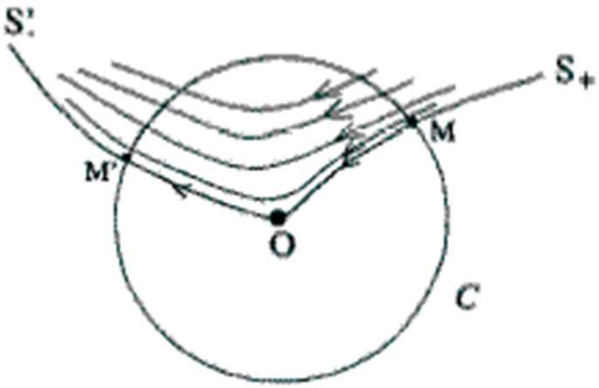


Figure 6: Hyperbolic sector

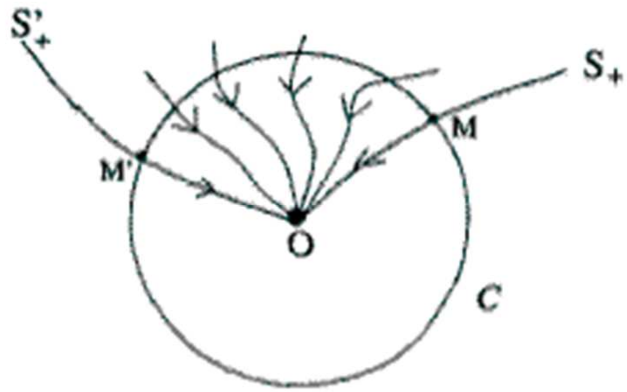


Figure 7: Parabolic sector

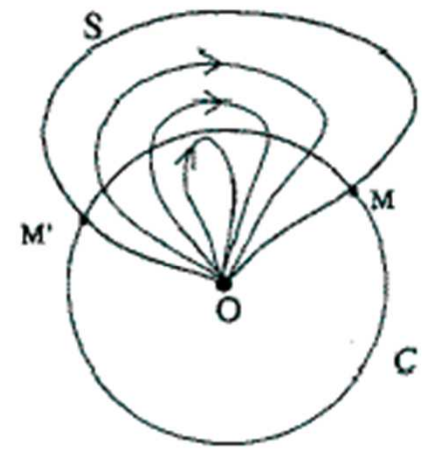


Figure 8: Elliptic sector

Source: A topology simplification method for 2D vector fields. Xavier Tricoche, Gerik Scheuermann, & Hans Hagen

# Sectors & Separatrices

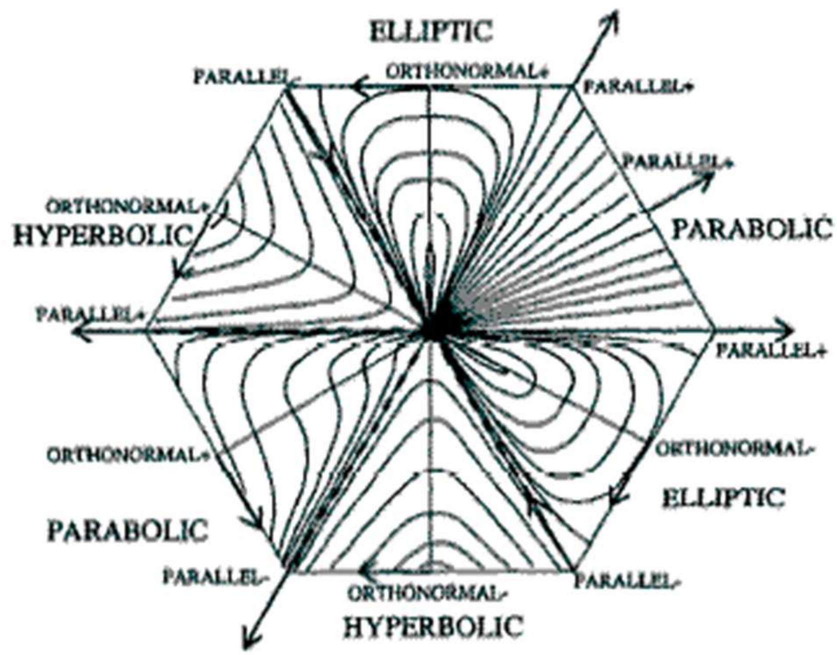
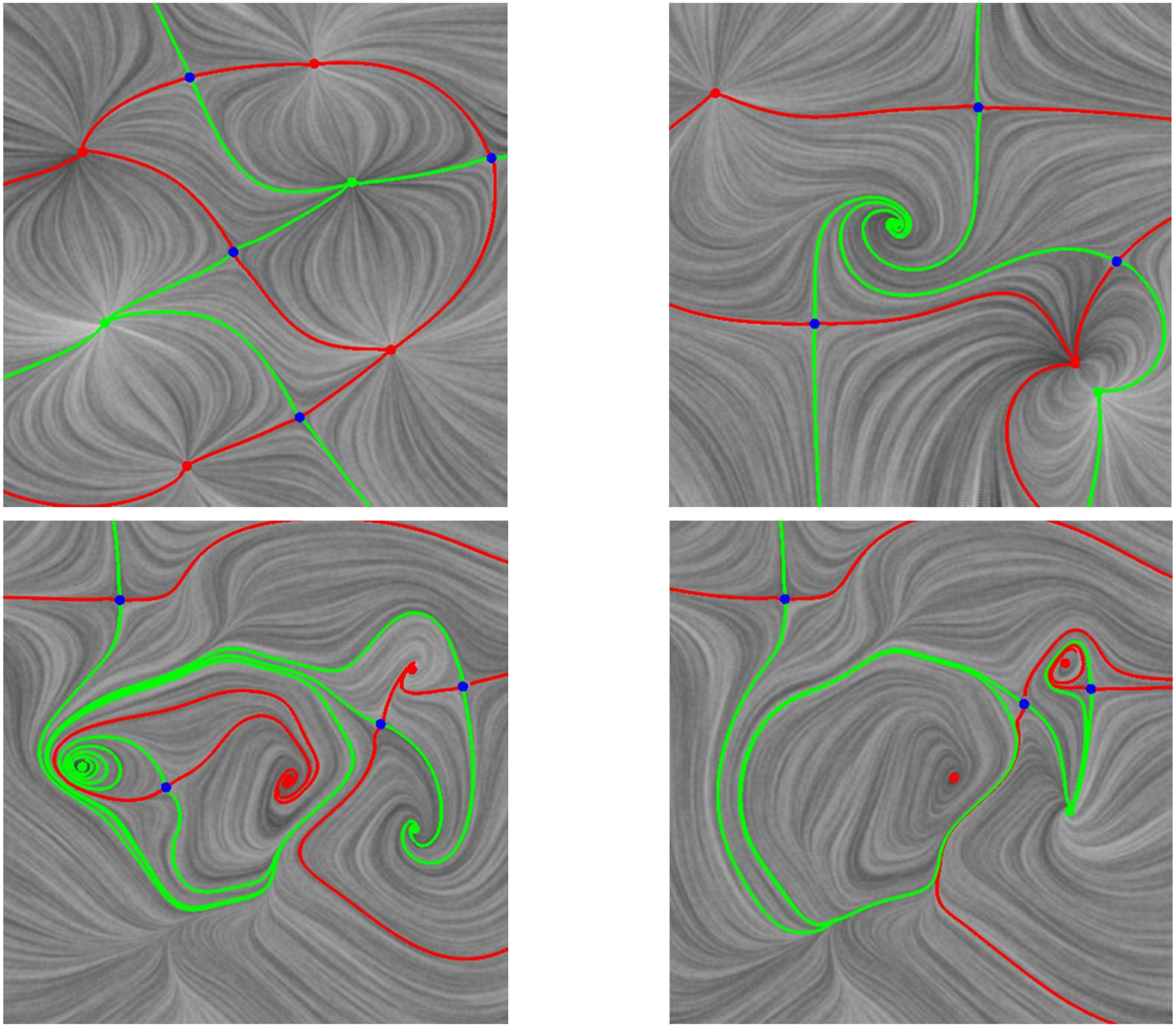


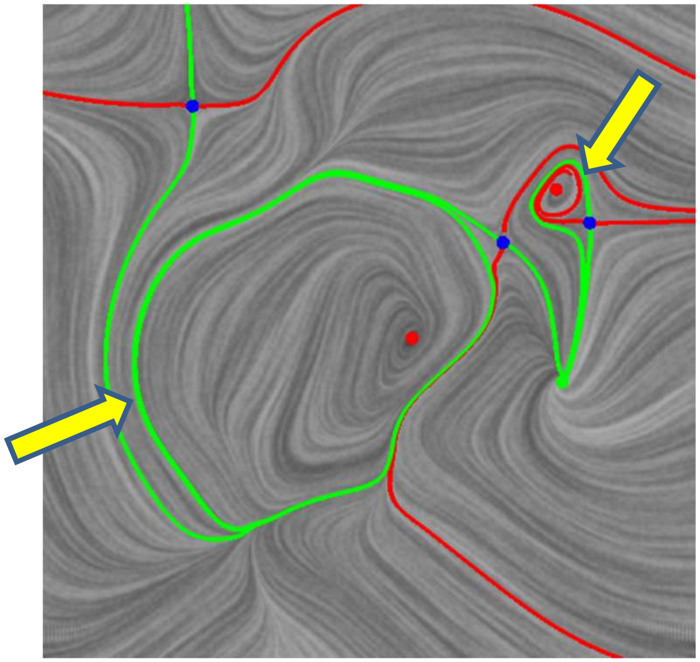

Figure 11: Example of sector type identification

Source: A topology simplification method for 2D vector fields. Xavier Tricoche, Gerik Scheuermann, & Hans Hagen

$$I = 1 + \frac{e - h}{2}$$

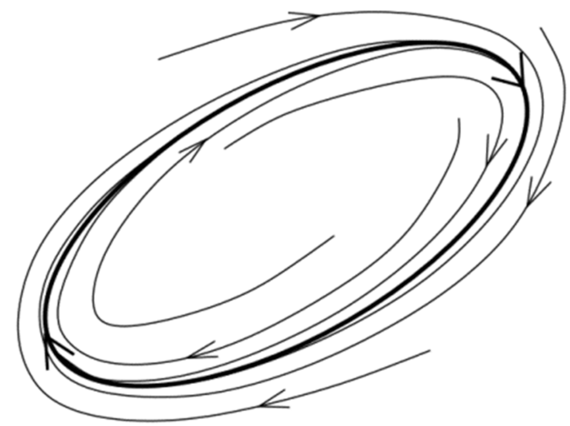


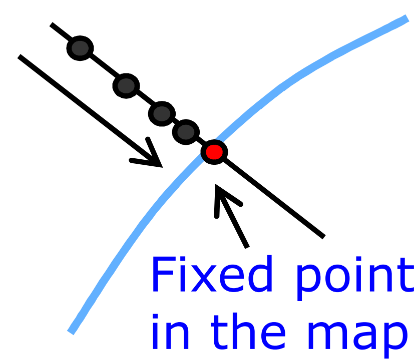
# What is missing?



# Periodic Orbits

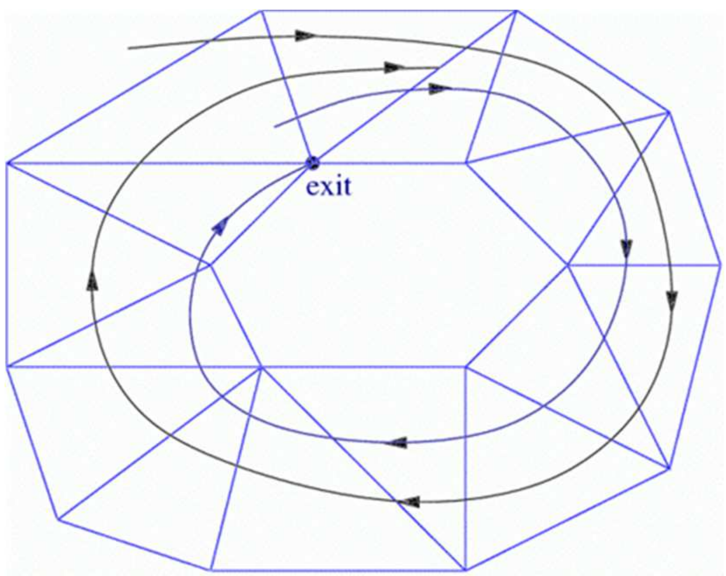
- Curve-type (1D) limit set
- Attracting / repelling behavior
- **Poincaré map**:
  - Defined over cross section
  - Map each position to next intersection with cross section along flow
  - Discrete map
  - Cycle intersects at fixed point
  - Hyperbolic / non-hyperbolic

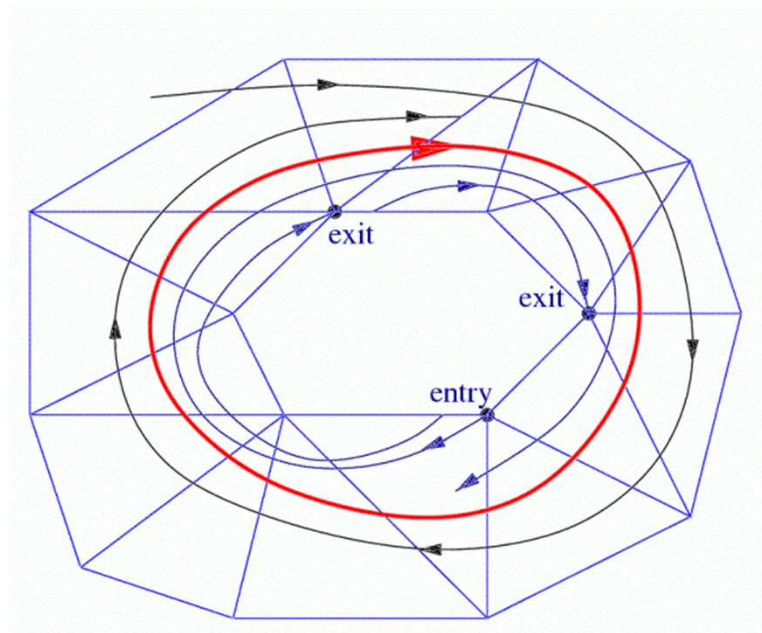




# Periodic Orbit Extraction

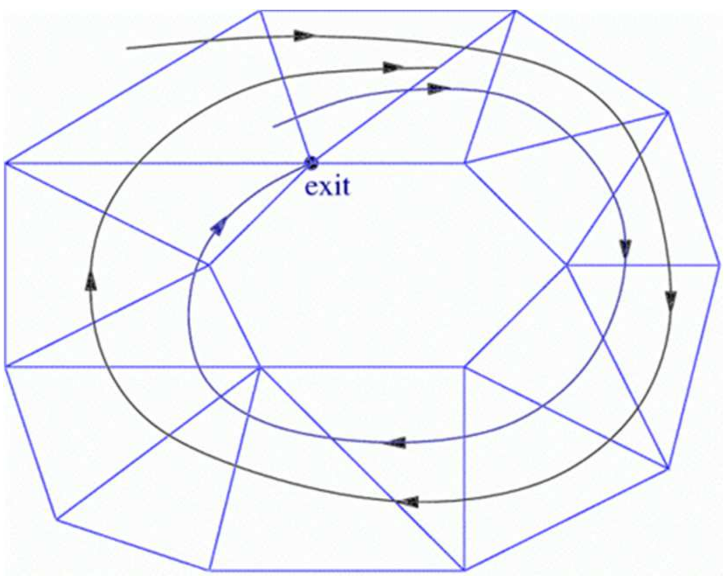
- Poincaré-Bendixson theorem:
  - If a region contains a limit set and no critical point, it contains a closed orbit

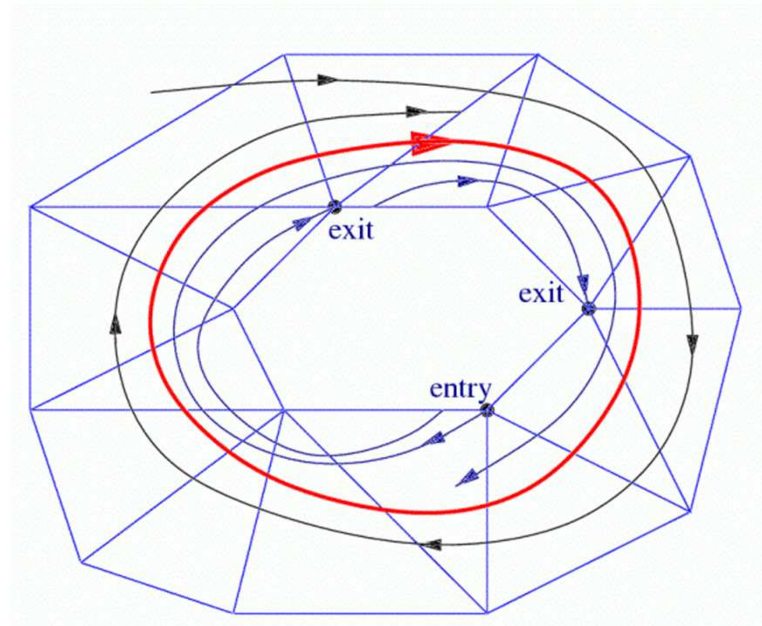


# Periodic Orbit Extraction

- Detect closed cell cycle
- Check for flow exit along boundary
- Find exact position with Poincaré map(fixed point)



**What is the** Poincaré Index of a periodic orbit?

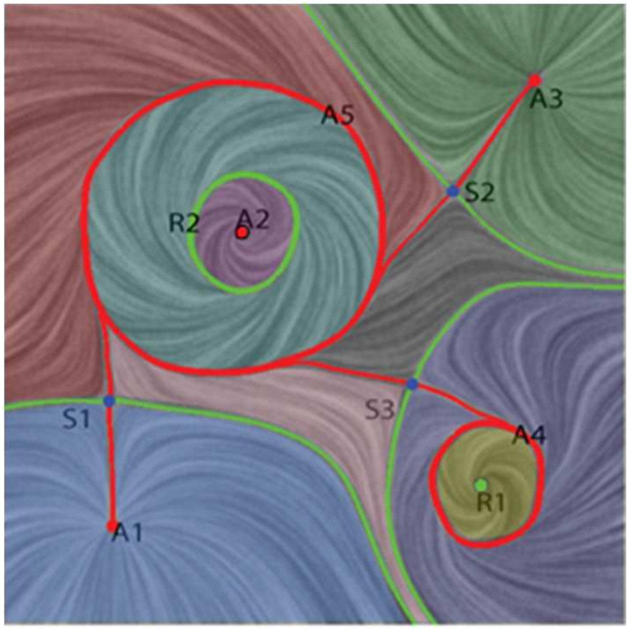
**What is the** Poincaré Index of a periodic orbit?

It is zero!

# Vector Field Topology - ECG

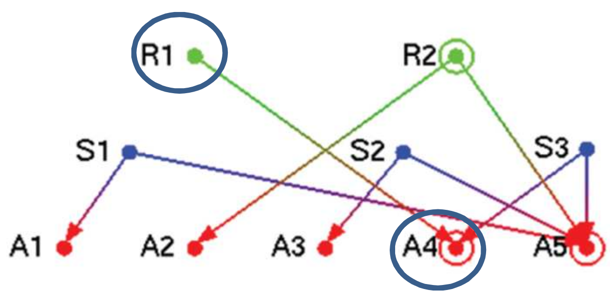
- Vector field topology provides qualitative (structural) information of the underlying dynamics

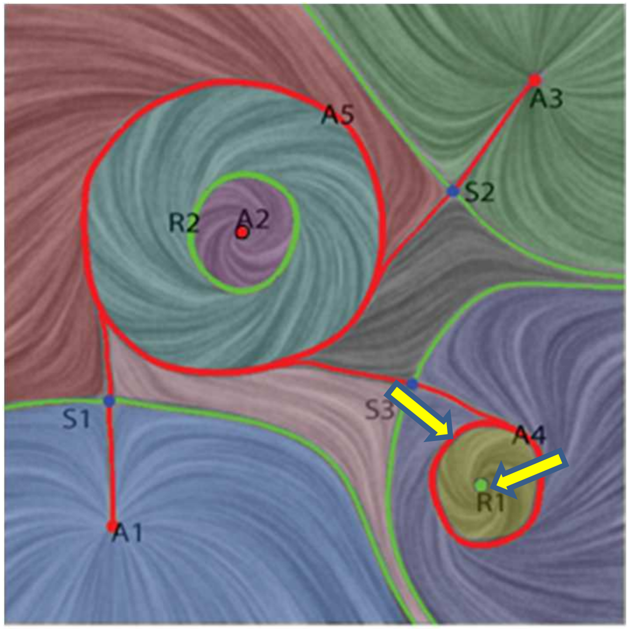
- It usually consists of certain critical features and their connectivity, which can be expressed as a graph, e.g. vector field skeleton [Helman and Hesselink 1989]
  - Fixed points
  - Periodic orbits
  - Separatrices



# Topological Graph - ECG

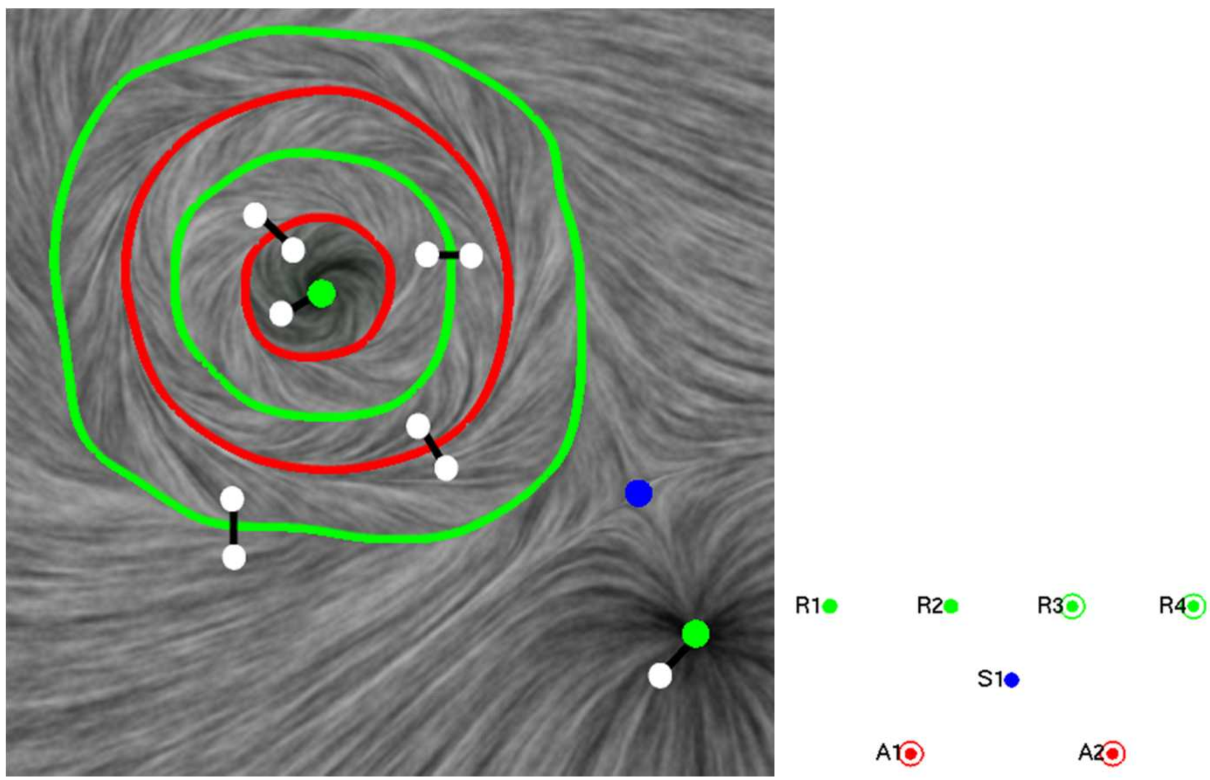
- Three layers based on the Conley index
  - Bottom (A)ttractors: sinks, attracting periodic orbits
  - Top (R)epellers: sources, repelling periodic orbits
  - Middle (S)addles



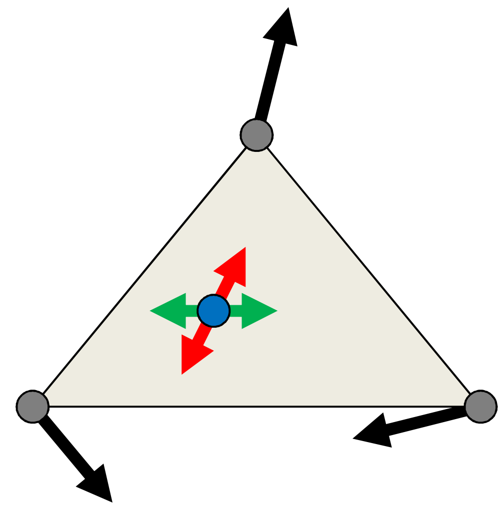
# Compute Connections

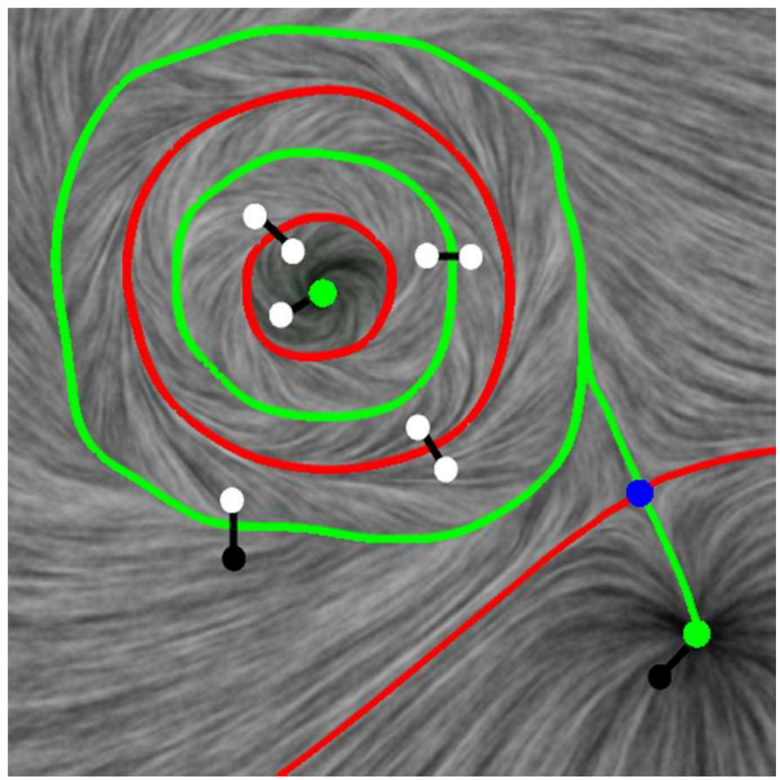
- Fixed point and periodic orbit extraction

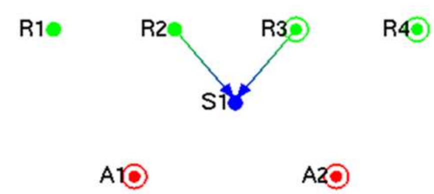


# Compute Connections

- Separatrices computation

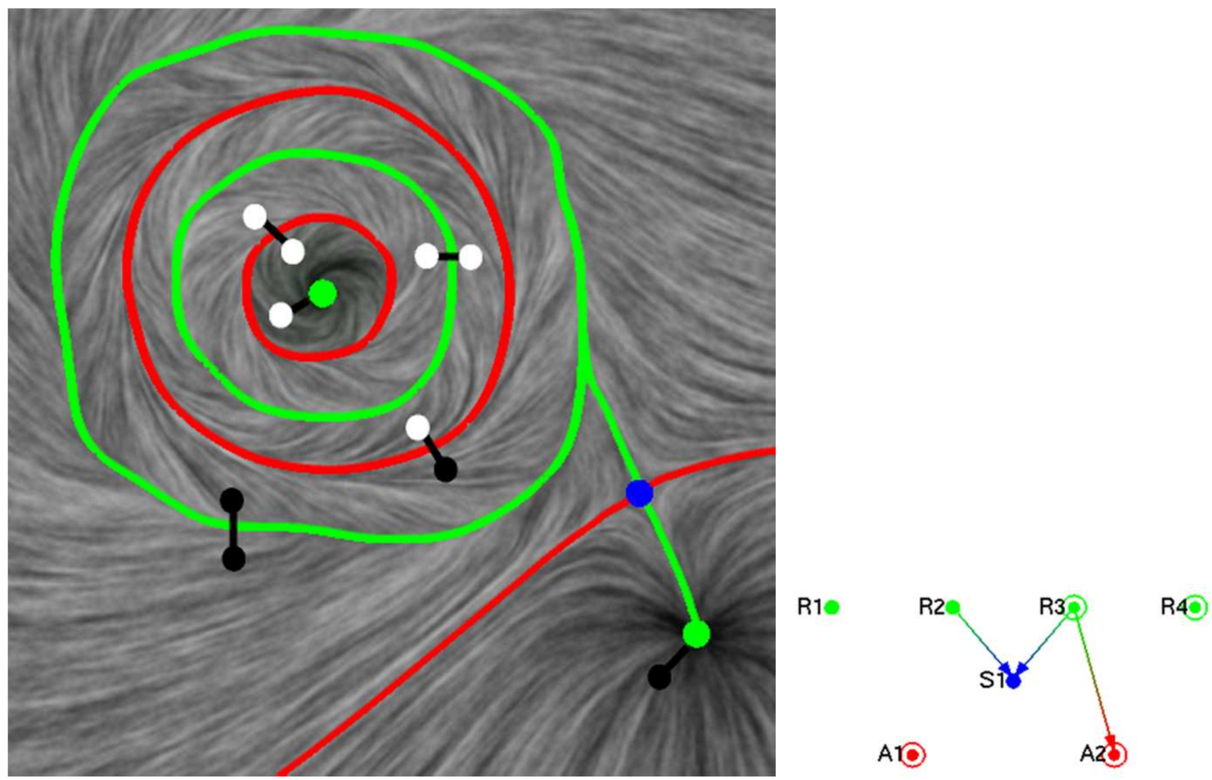






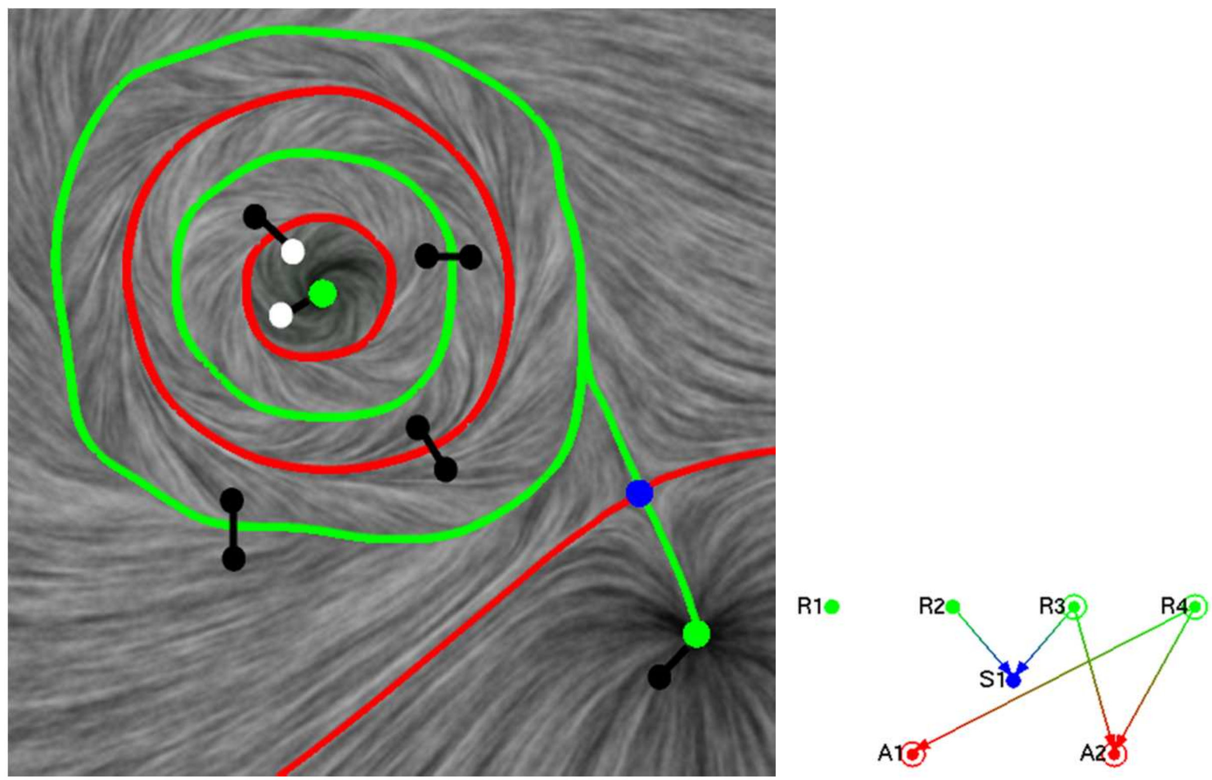
# Compute Connections

- Repelling periodic orbit to a fixed point or orbit



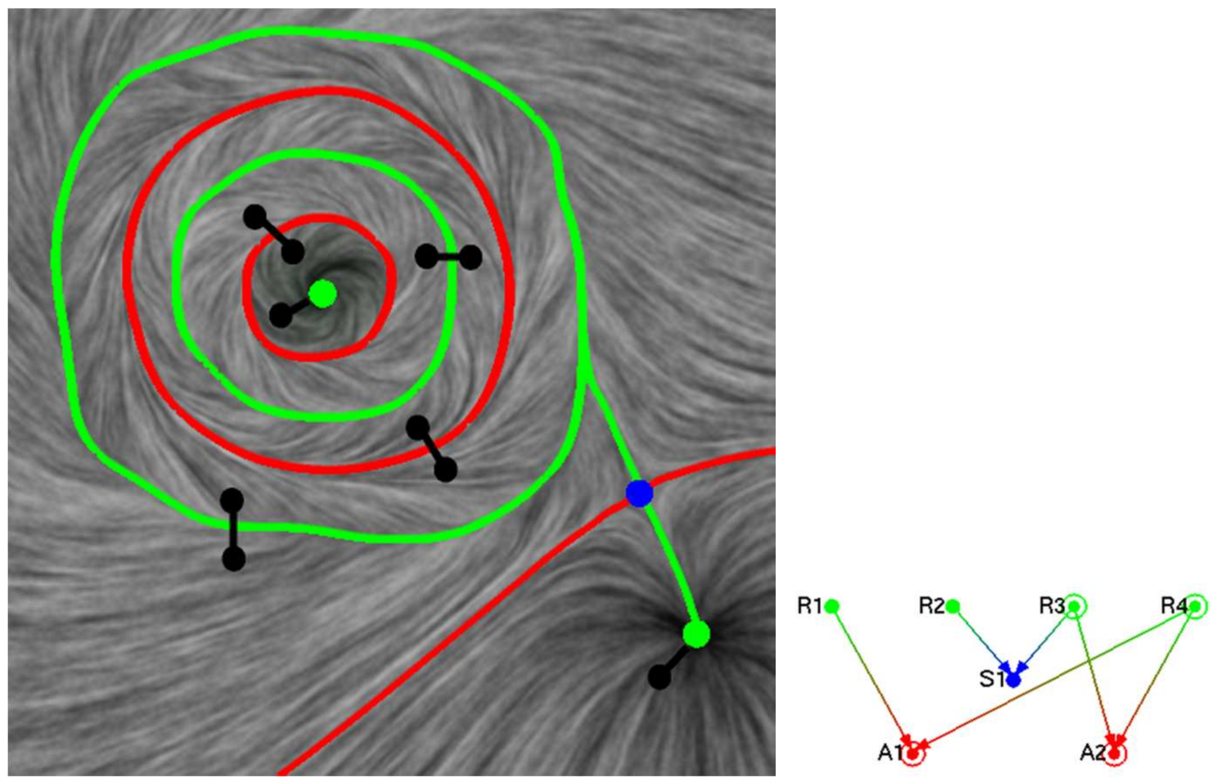
# Compute Connections

- Repelling periodic orbit to a fixed point or orbit



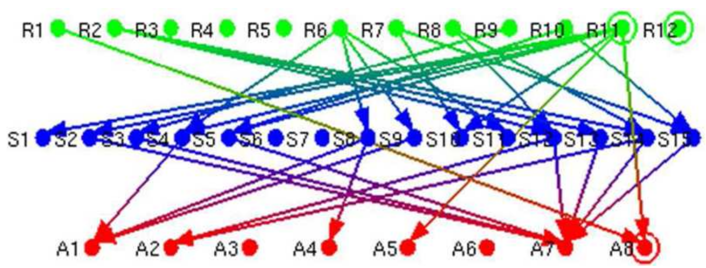
# Compute Connections

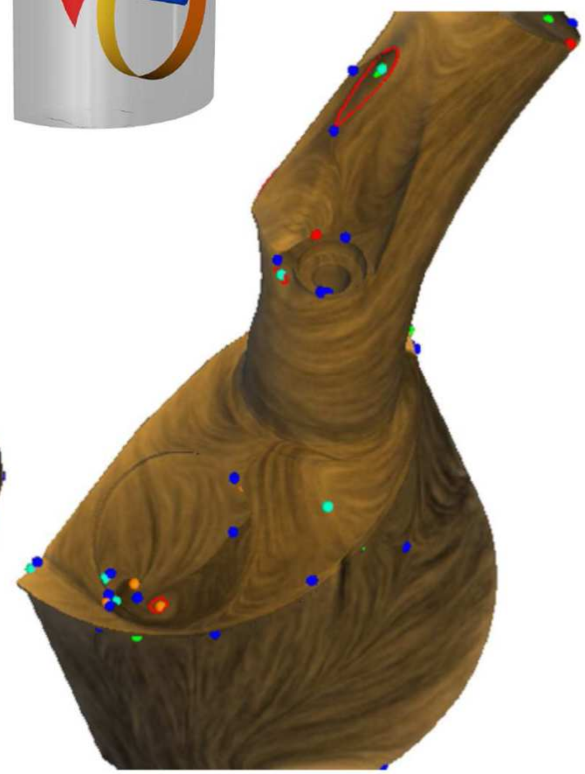
- Attracting periodic orbit to a fixed point



# Applications (1)

- CFD simulation on gas engine
- Velocity extrapolated to the boundary



105K polygons

56 fixed points

9 periodic orbits

31.58s on analysis

# Applications (2)

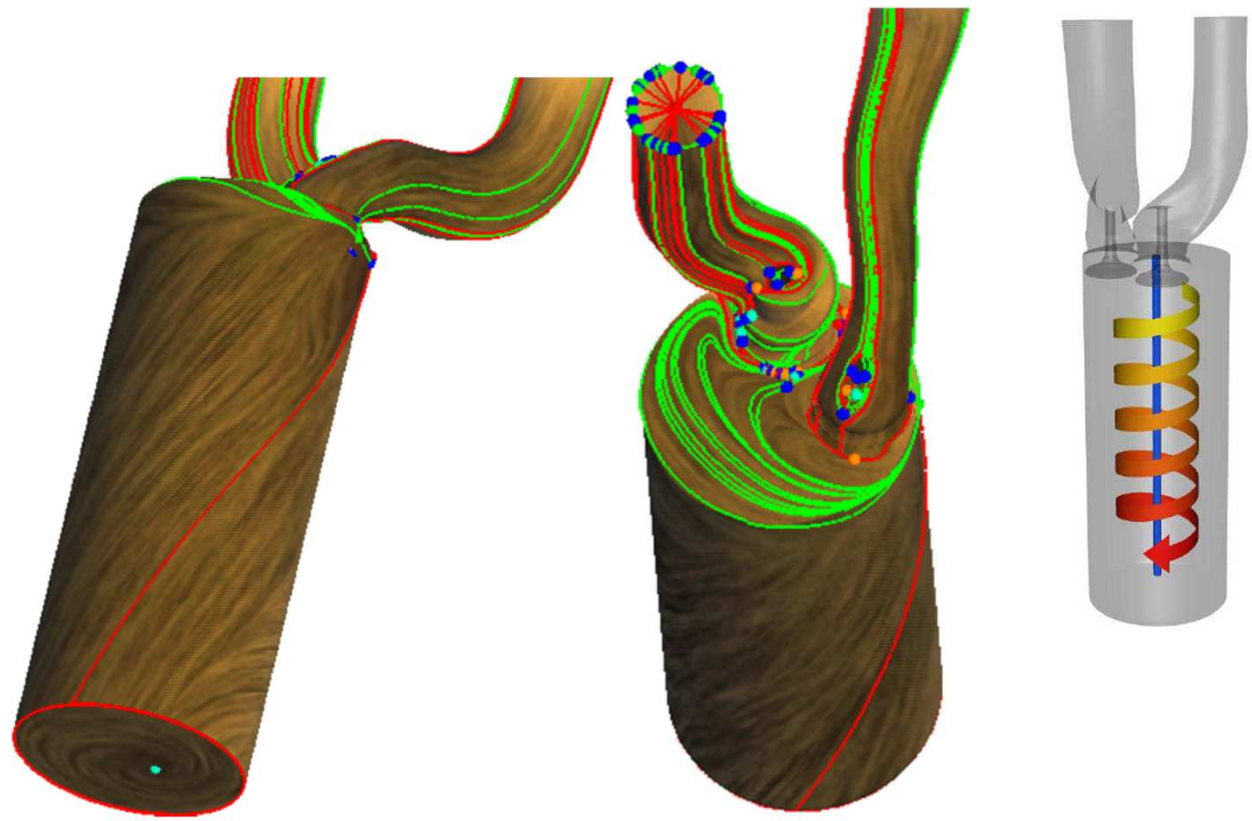
- CFD simulation on diesel engine
- Velocity extrapolated to the boundary

886K polygons

226 fixed points

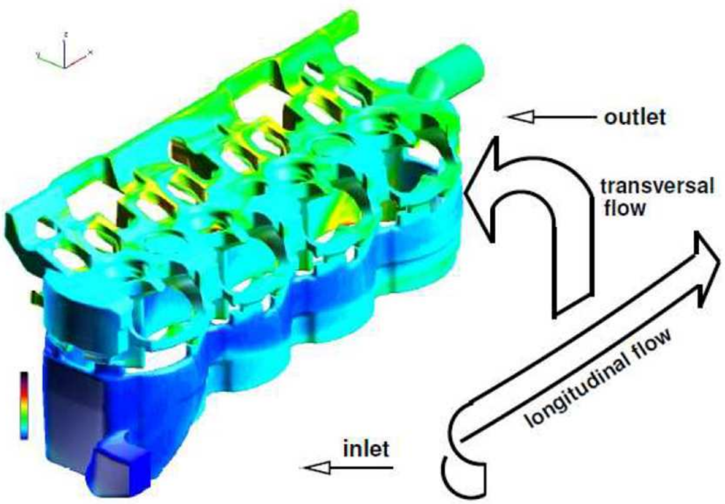
52 periodic orbits

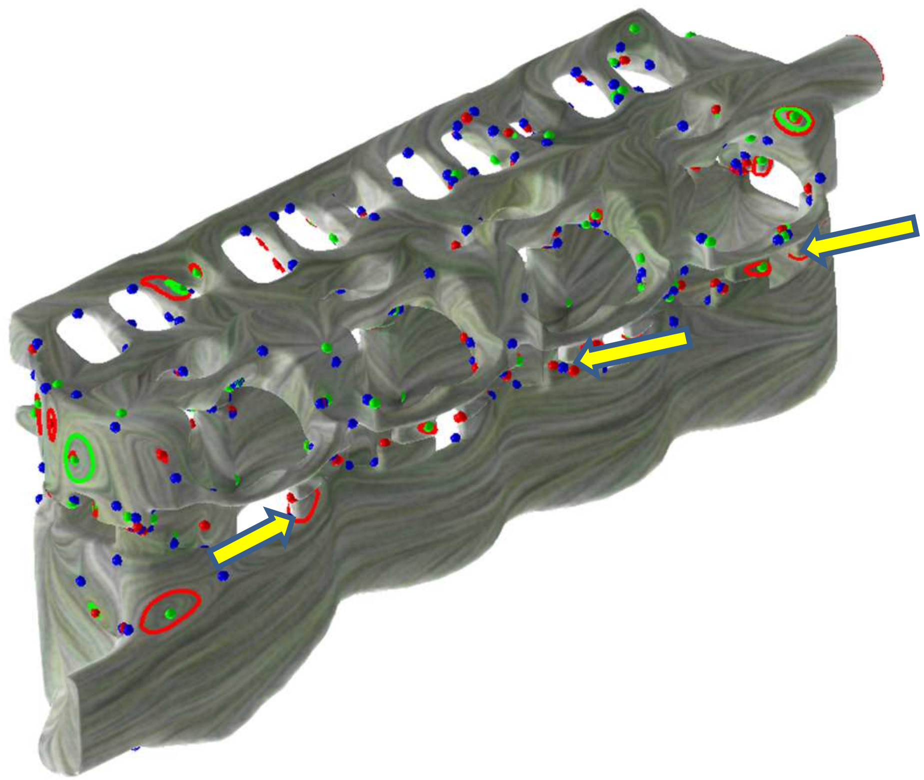
29.15s on analysis



# Application (3)

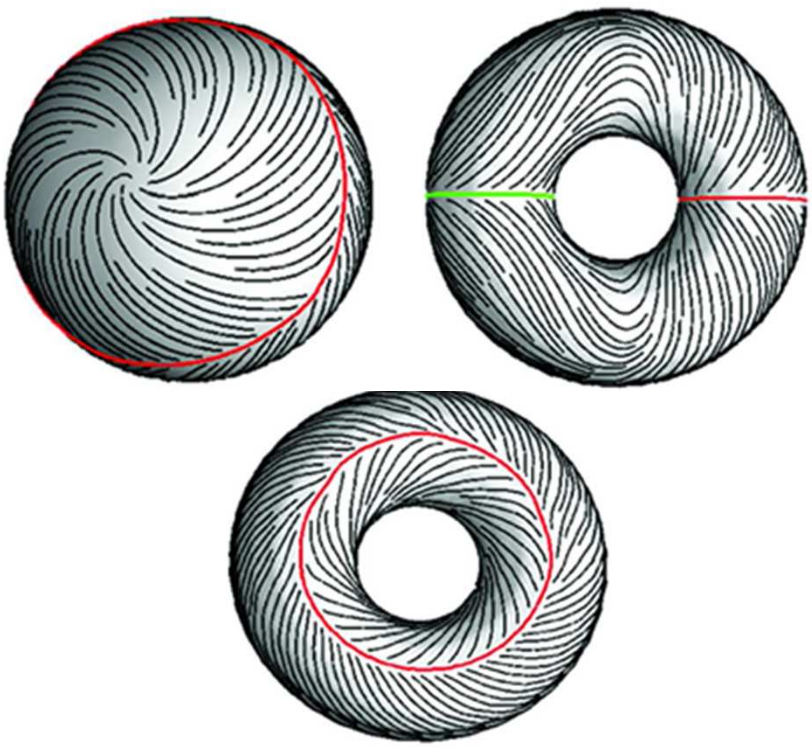
- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary

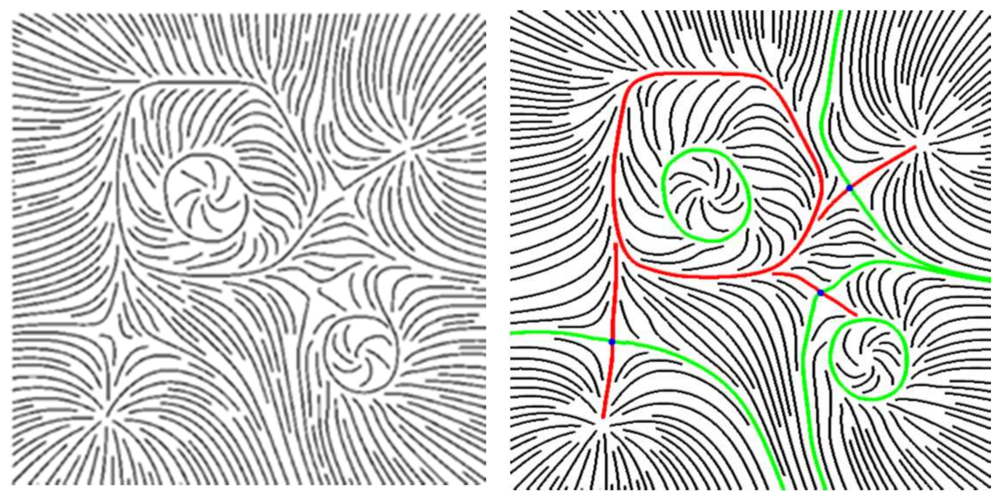


# Applications (4)

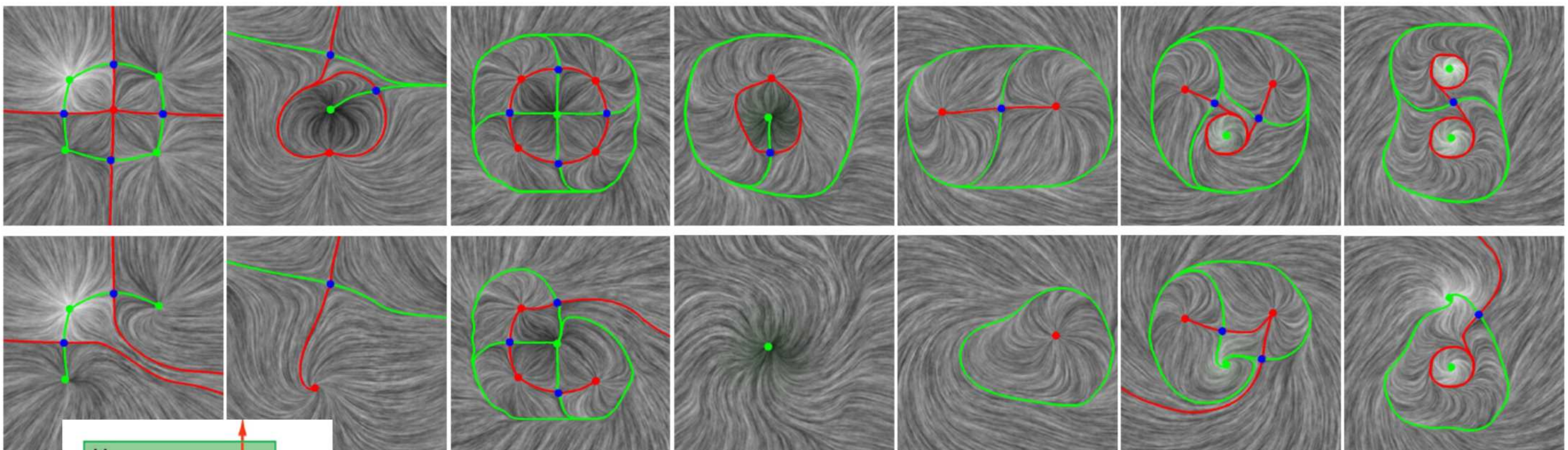
- Feature-aware streamline placement
  - First extract topology, then use it as the initial set of streamlines to compute seeds for later placement

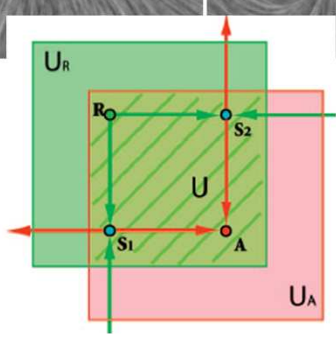




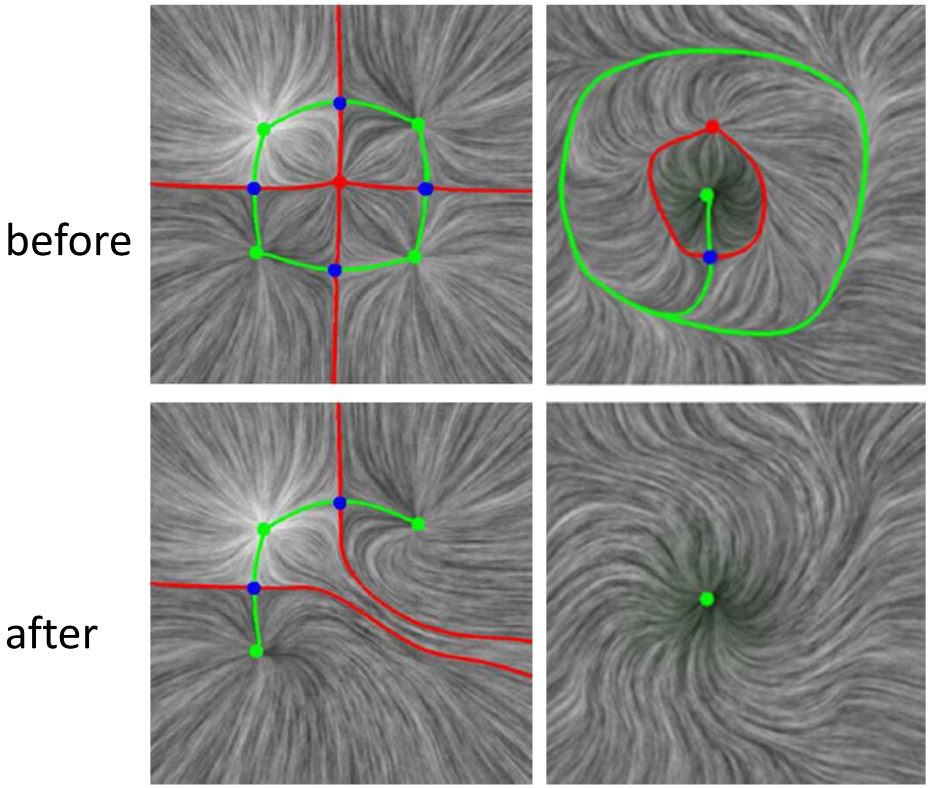
# Simplification

Reduce flow complexity so that people can focus on the more important structure

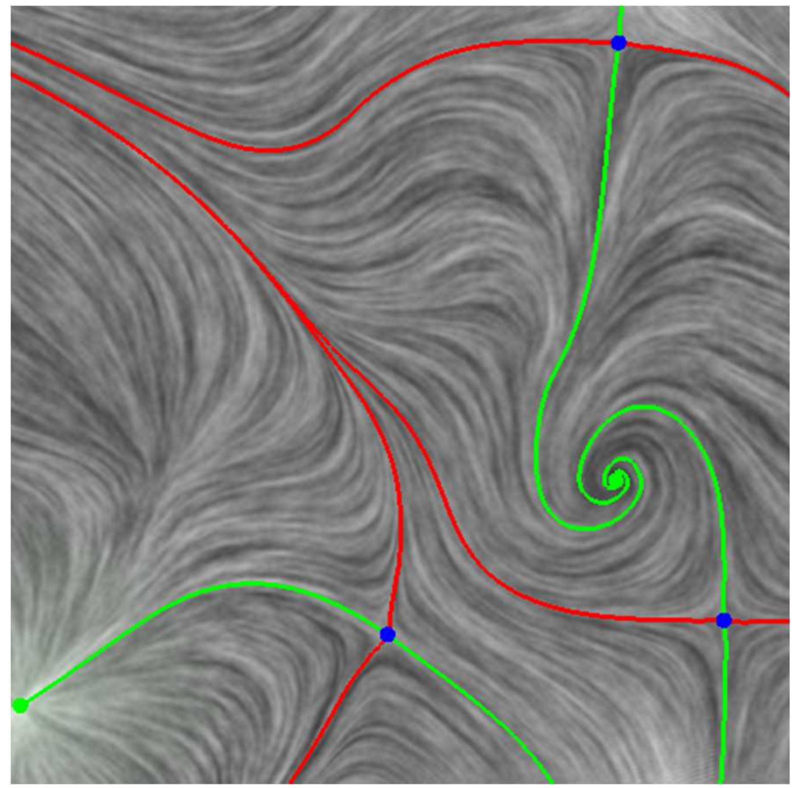




# Vector Field Editing



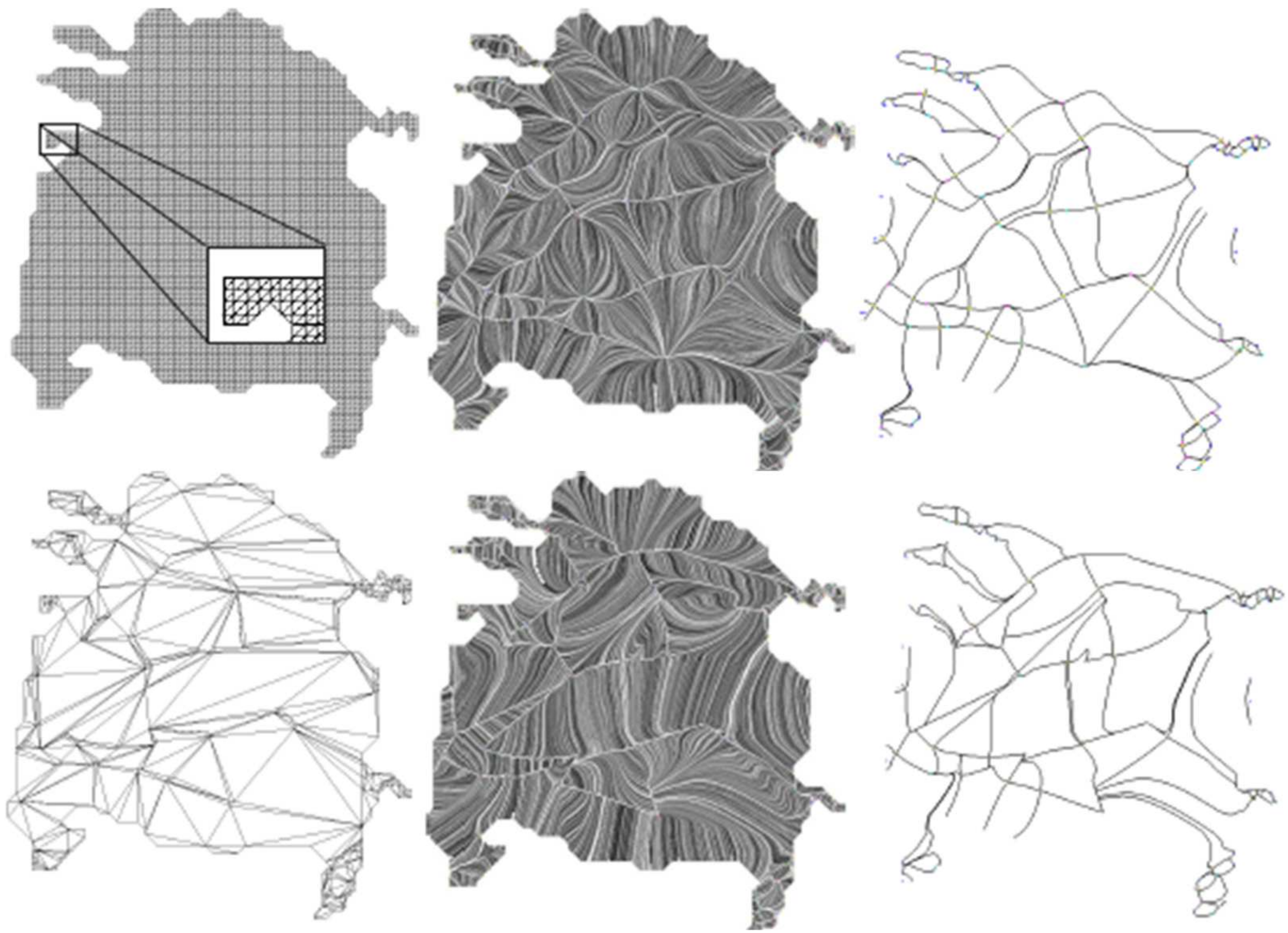

Simplification



Relocation

**Based on ECG**

# Vector Field Data Compression



Source: [Theisel et al. Eurographics 2003]

# Additional Reading

- Frits H. Post, Benjamin Vrolijk, Helwig Hauser, Robert S. Laramee and Helmut Doleisch. **The State of the Art in Flow Visualization: Feature Extraction and Tracking**. Computer Graphics Forum, 22 (4): pp. 1-17, 2003.
- Robert S. Laramee, Helwig Hauser, Lingxiao Zhao, and Frits H. Post, **Topology-Based Flow Visualization, The State of the Art**, in *Topology-Based Methods in Visualization (Proceedings of Topo-In-Vis 2005, 29–30 September 2005, Budmerice, Slovakia)*, Mathematics and Visualization, H. Hauser, H. Hagen, and H. Theisel editors, pages 1-19, 2007, Springer-Verlag.
- Guoning Chen, Konstantin Mischaikow, Robert S. Laramee, Pawel Pilarczyk, and Eugene Zhang. **Vector Field Editing and Periodic Orbit Extraction Using Morse Decomposition**. IEEE Transactions on Visualization and Computer Graphics, Vol. 13, No. 4, 2007, pp. 769-785.

# Acknowledgment

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