Chapter 2

Number Systems, Arithmetic, and Code

Positional number systems

- What is the underlying principle?
- Can you find an example of a number system that is not positional?
- What are the reasons for using different bases?

Notational convention

- 234.16

  $= 2 \times 100 + 3 \times 10 + 4 \times 1 + 1 \times 0.1 + 6 \times 0.01$

  $= 2 \times 10^{2} + 3 \times 10^{1} + 4 \times 10^{0} + 1 \times 10^{-1} + 6 \times 10^{-2}$

Basic arithmetic operations

- The basic operations are addition, subtraction, multiplication, and division.
- They are very similar for positional number systems with different bases.
- Solutions to any computational problems to be solved on a computer must be expressed in terms of these operations.

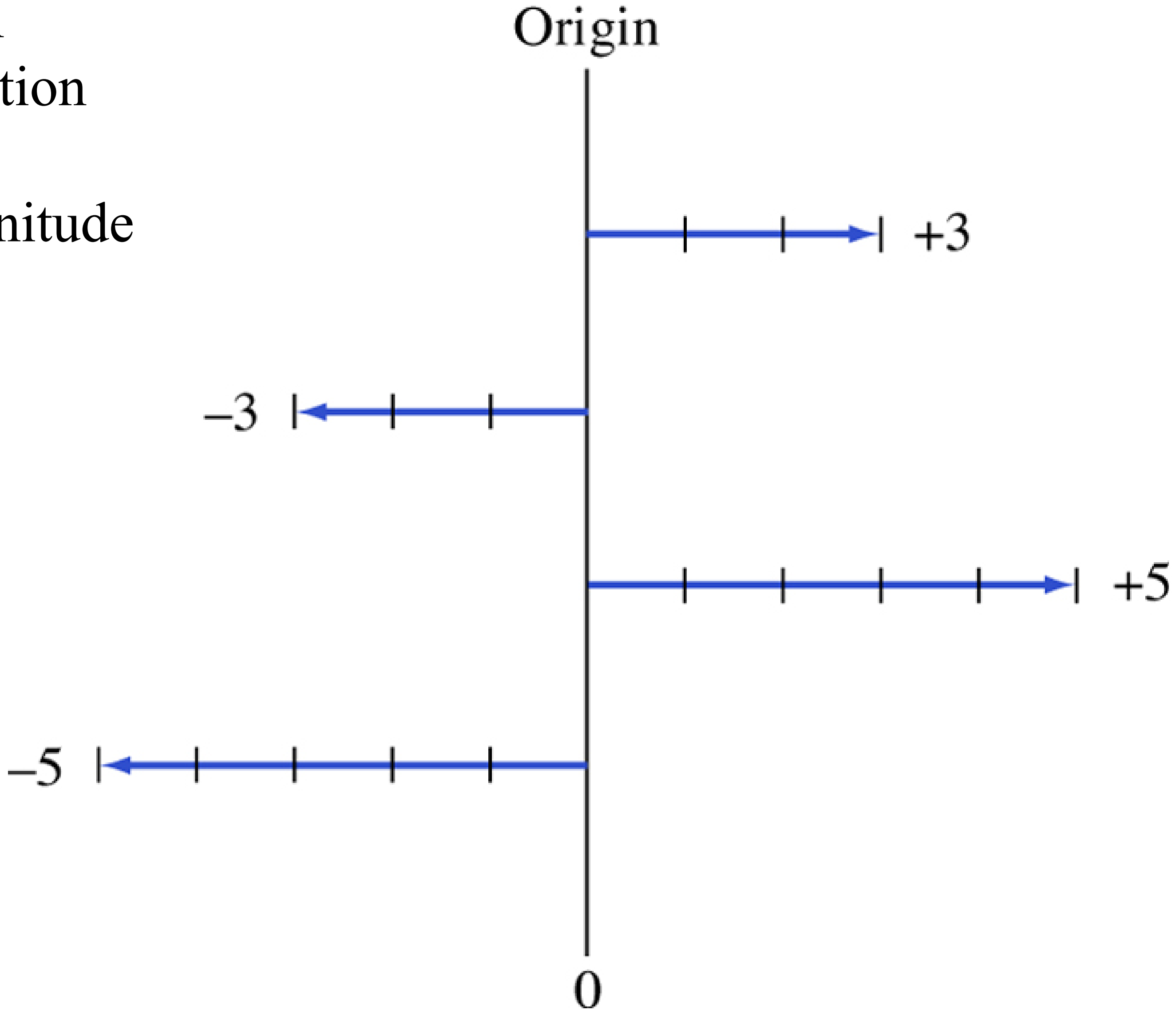
Methods of number conversion

- Polynomial method
- Iterative method
- Special conversion method

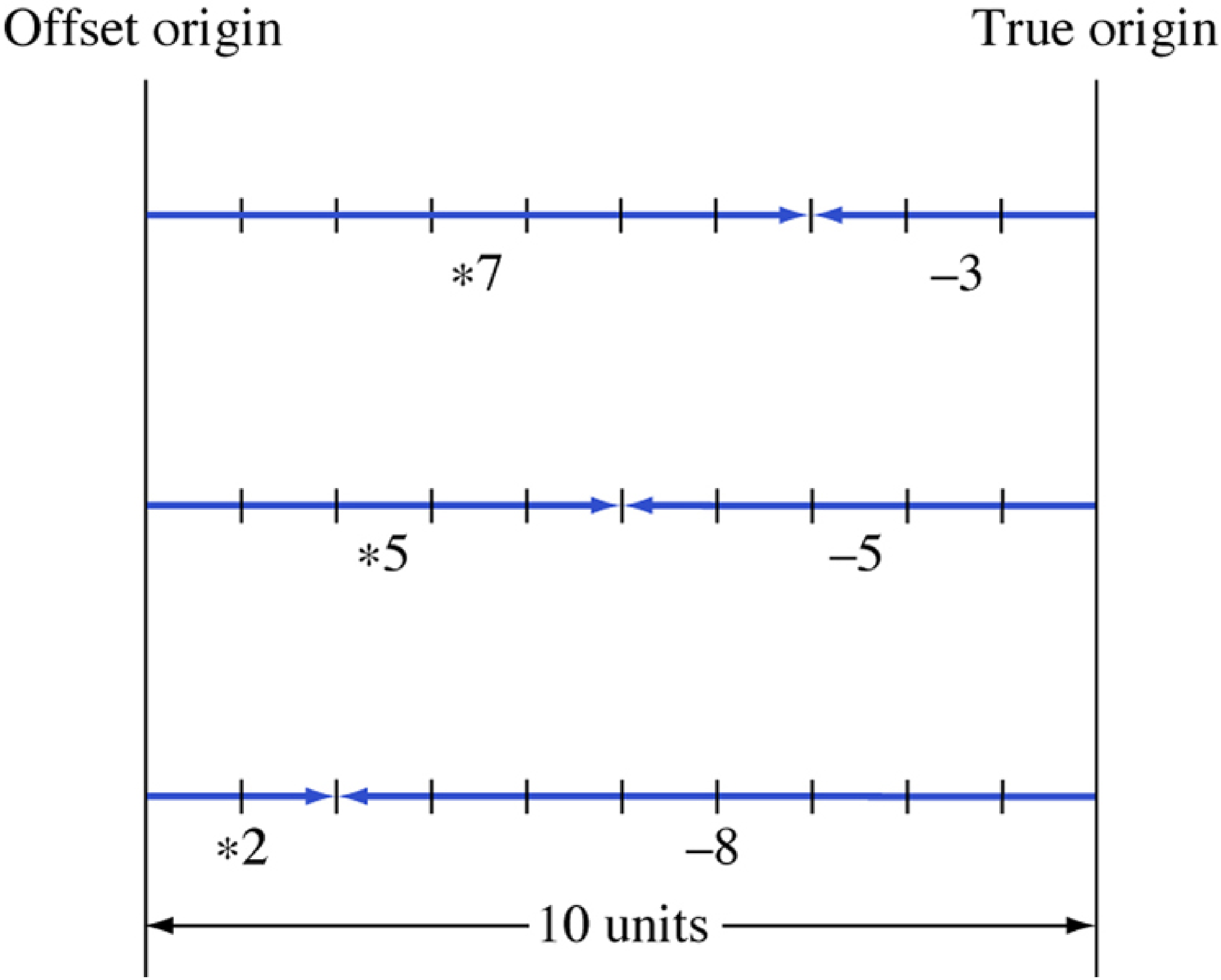
Signed numbers

- sign and magnitude
- r’s-complement
- (r-1)’s complement

Graphical interpretation of sign-magnitude numbers



Graphical interpretation of complements



Subtraction

- Shown below are the steps involved in performing M-N through complementation and addition.
- Here M and N are unsigned numbers.

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
Definition: given N with n digits	$(r^n - 1) - N$	$r^n - N$

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
Computation involved in obtaining the complement	digitwise (bitwise) complementation	digitwise (bitwise) complementation + 1

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
Number of zero	two (+0 and -0)	one

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
Subtraction operation	$M - N \rightarrow M + (r^n-1) - N$ $= r^n + M - N - 1$ $= (r^n - 1) - (N - M)$	$M - N \rightarrow M + r^n - N$ $= r^n + M - N$ $= r^n - (N - M)$

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
If M > N	M-N → r^n + M - N - 1 *What will happen?* There is a carry. *How to produce M-N?* (1) Subtract r^n by discarding the carry. (2) Add 1 to it.	M - N → M + r^n - N *What will happen?* There is a carry. *How to produce M-N?* Subtract r^n by discarding the carry.

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
if M < N	M - N → $(r^n-1) - (N - M)$ *What will happen?* There is no carry. *How to produce M-N (in sign-and-magnitude)?* The result is negative and in (r-1)'s complement. (1) Perform (r-1)'s complement to obtain (N-M). (2) Prefix it with a minus sign to indicate that it is negative.	M - N → $r^n - (N-M)$ *What will happen?* There is no carry. *How to produce M-N (in sign-and-magnitude)?* The result is negative and in r's complement. (1) Perform r's complement (i.e., (r-1)'s complement plus 1) to obtain (N-M). (2) Prefix it with a minus sign to indicate that it is negative.

(r-1)’s *vs*. r’s complement

	(r-1)'s complement	r's complement
if M = N	$M\text{-}N \rightarrow r^n + M - N - 1$ *What will happen?* There is no carry. *How to produce M-N?* It is treated as if M<N, producing a "-0" as the result.	$M - N \rightarrow M + r^n - N$ *What will happen*? There is a carry. *How to produce M-N?* It is treated as if M>N, producing a "0" as the result.

Interpretation of four-bit signed binary integers

$b_3b_2b_1b_0$	sign and mag.	1's complement	2's complement
0111	**+7**	**+7**	**+7**
0110	**+6**	**+6**	**+6**
0101	**+5**	**+5**	**+5**
0100	**+4**	**+4**	**+4**
0011	**+3**	**+3**	**+3**
0010	**+2**	**+2**	**+2**
0001	**+1**	**+1**	**+1**
0000	**+0**	**+0**	**+0**
1000	**-0**	**-7**	**-8**
1001	**-1**	**-6**	**-7**
1010	**-2**	**-5**	**-6**
1011	**-3**	**-4**	**-5**
1100	**-4**	**-3**	**-4**
1101	**-5**	**-2**	**-3**
1110	**-6**	**-1**	**-2**
1111	**-7**	**-0**	**-1**

The use of 2’s complement

- In practice, signed numbers are always represented by 2’s complements because then there is only one zero.
- Existence of more than one zero leads to complication in programming.

$$\begin{array}{r} (+5) \\ +(+2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\ 1\ 0\ 1 \\ +\ 0\ 0\ 1\ 0 \\ \hline 0\ 1\ 1\ 1 \end{array} \qquad\qquad \begin{array}{r} (-5) \\ +(+2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 0 \\ +\ 0\ 0\ 1\ 0 \\ \hline 1\ 1\ 0\ 0 \end{array}$$

$$\begin{array}{r} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\ 1\ 0\ 1 \\ +\ 1\ 1\ 0\ 1 \\ \hline 1\ 0\ 0\ 1\ 0 \\ \llcorner\!\rightarrow \quad 1 \\ \hline 0\ 0\ 1\ 1 \end{array} \qquad\qquad \begin{array}{r} (-5) \\ +(-2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 0 \\ +\ 1\ 1\ 0\ 1 \\ \hline 1\ 0\ 1\ 1\ 1 \\ \llcorner\!\rightarrow \quad 1 \\ \hline 1\ 0\ 0\ 0 \end{array}$$

Examples of 1's complement addition

$$\begin{array}{r} (+5) \\ +\ (+2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\ 1\ 0\ 1 \\ +\ 0\ 0\ 1\ 0 \\ \hline 0\ 1\ 1\ 1 \end{array} \qquad\qquad \begin{array}{r} (-5) \\ +\ (+2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 1 \\ +\ 0\ 0\ 1\ 0 \\ \hline 1\ 1\ 0\ 1 \end{array}$$

$$\begin{array}{r} (+5) \\ +\ (-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\ 1\ 0\ 1 \\ +\ 1\ 1\ 1\ 0 \\ \hline \underset{\overset{\uparrow}{\text{ignore}}}{1}\ 0\ 0\ 1\ 1 \end{array} \qquad\qquad \begin{array}{r} (-5) \\ +\ (-2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 1 \\ +\ 1\ 1\ 1\ 0 \\ \hline \underset{\overset{\uparrow}{\text{ignore}}}{1}\ 1\ 0\ 0\ 1 \end{array}$$

Examples of 2's complement addition

$$\begin{array}{r} (+5) \\ -\,(+2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\,0\,0\,1\,0 \\ \hline \\ \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,1\,0 \\ \hline \underset{\substack{\uparrow \\ \text{ignore}}}{1}\,0\,0\,1\,1 \end{array}$$

$$\begin{array}{r} (-5) \\ -\,(+2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\,0\,0\,1\,0 \\ \hline \\ \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 1\,0\,1\,1 \\ +\,1\,1\,1\,0 \\ \hline \underset{\substack{\uparrow \\ \text{ignore}}}{1}\,1\,0\,0\,1 \end{array}$$

$$\begin{array}{r} (+5) \\ -\,(-2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\,1\,1\,1\,0 \\ \hline \\ \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,0\,0\,1\,0 \\ \hline 0\,1\,1\,1 \end{array}$$

$$\begin{array}{r} (-5) \\ -\,(-2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\,1\,1\,1\,0 \\ \hline \\ \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 1\,0\,1\,1 \\ +\,0\,0\,1\,0 \\ \hline 1\,1\,0\,1 \end{array}$$

Examples of 2's complement subtraction

Overflow

- In adding two binary numbers, an *overflow* condition is said to occur if the resulting sum requires more bits than are available.
- Let x be the carry into the sign-bit position and y be the carry from the sign-bit position, then there is an overflow if and only if $x \oplus y = 1$.

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array} \qquad \begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$$c_4 = 0 \qquad\qquad c_4 = 0$$
$$c_3 = 1 \qquad\qquad c_3 = 0$$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array} \qquad \begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

$$c_4 = 1 \qquad\qquad c_4 = 1$$
$$c_3 = 1 \qquad\qquad c_3 = 0$$

There is an overflow if $c_3 \oplus c_4 = 1$

Examples of determination of overflow

Decimal codes

- Weighted decimal codes
- Non-weighted decimal codes
- Bar codes

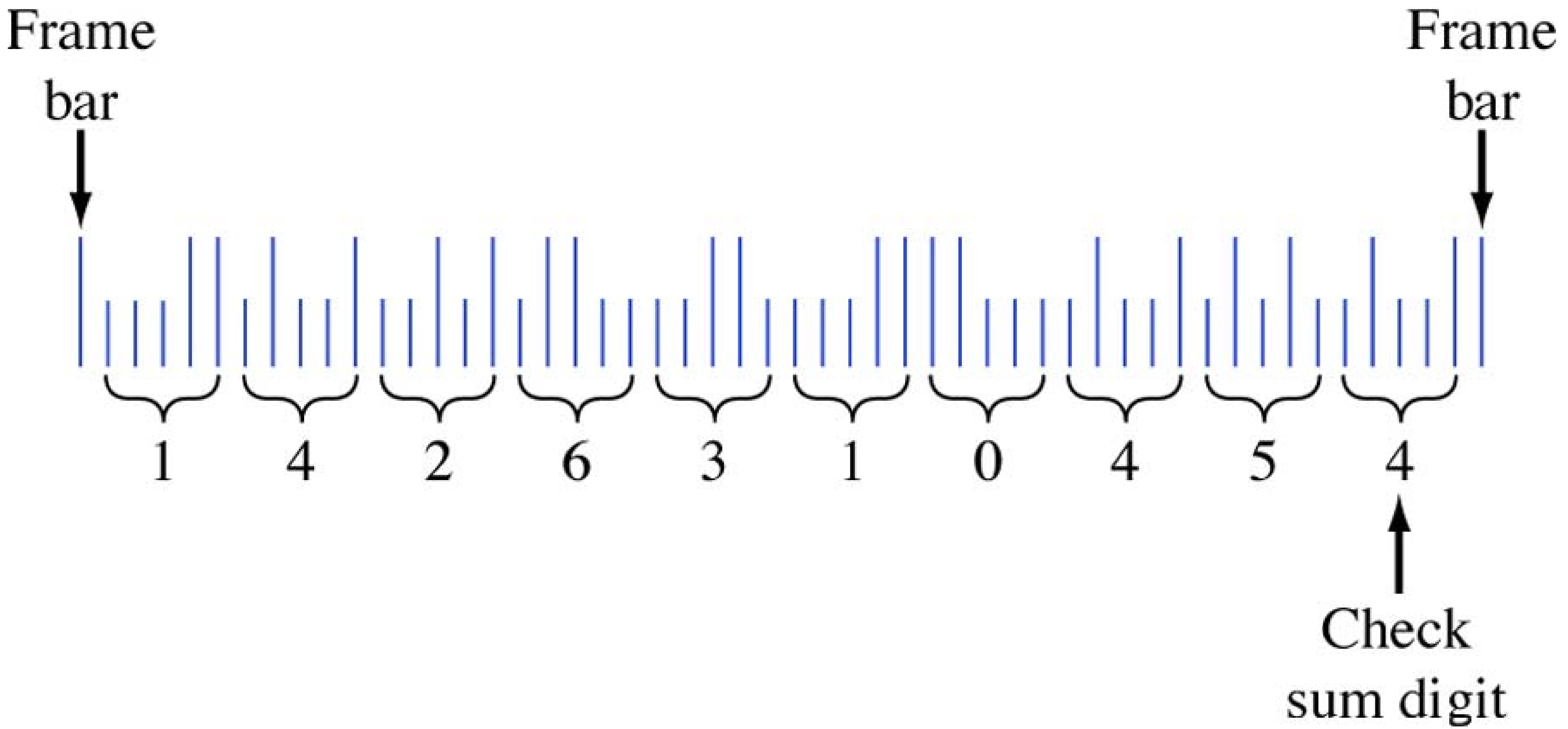
Weighted decimal codes

decimal digit	8421 code (BCD)	2421 code	5421 code	7536 code	biquinary code 5043210
0	0000	0000	0000	0000	0100001
1	0001	0001	0001	1001	0100010
2	0010	0010	0010	0111	0100100
3	0011	0011	0011	0010	0101000
4	0100	0100	0100	1011	0110000
5	0101	1011	1000	0100	1000001
6	0110	1100	1001	1101	1000010
7	0111	1101	1010	1000	1000100
8	1000	1110	1011	0110	1001000
9	1001	1111	1100	1111	1010000

Nonweighted decimal codes

decimal digit	excess-3 code	2-out-of-5 code
0	0011	11000
1	0100	00011
2	0101	00101
3	0110	00110
4	0111	01001
5	1000	01010
6	1001	01100
7	1010	10001
8	1011	10010
9	1100	10100

US Postal Service bar code (2 out of 5)



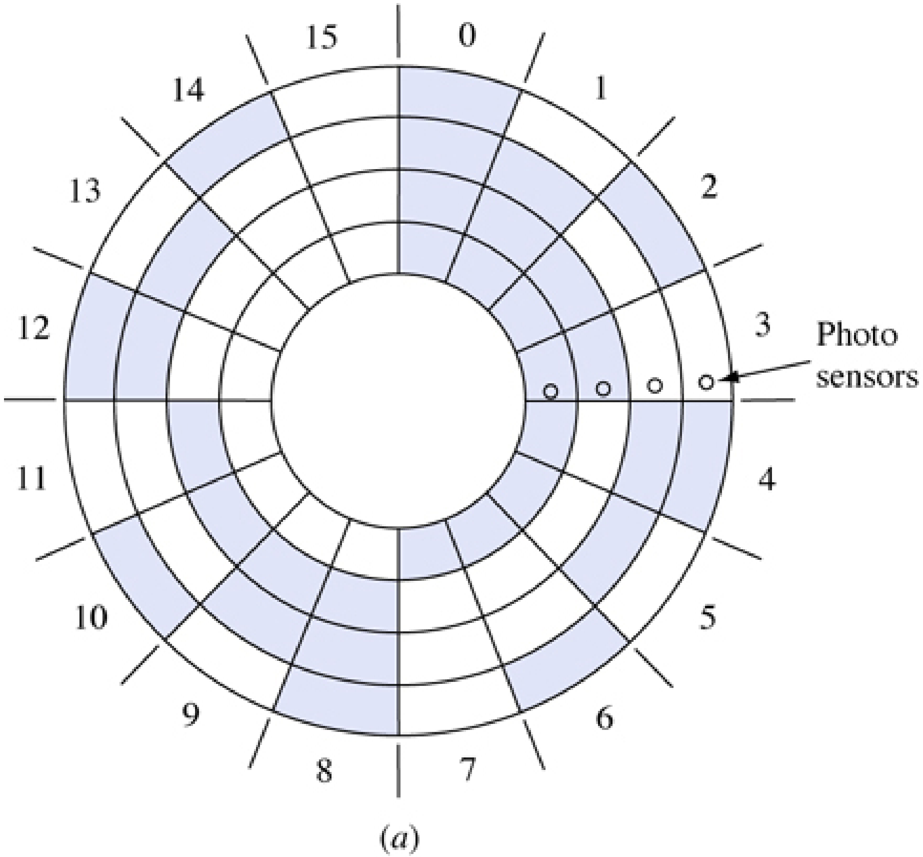
Gray code (a unit distance code)

decimal number	Gray code
0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100

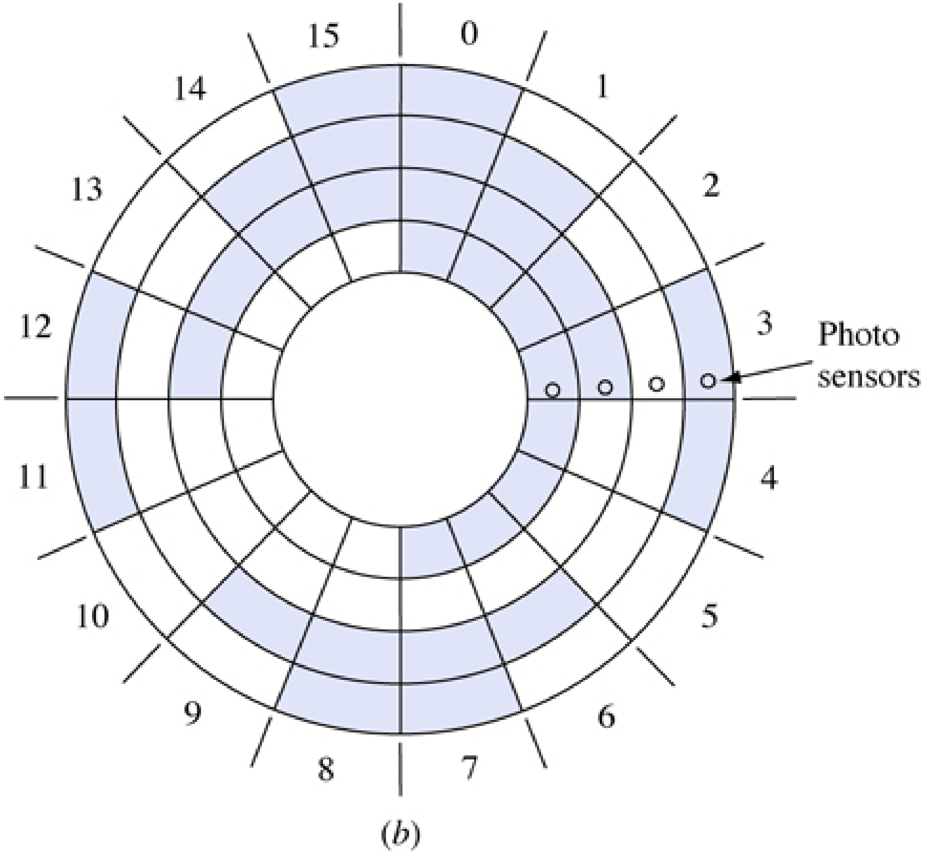
Angular position encoders

conventional binary

Gray code

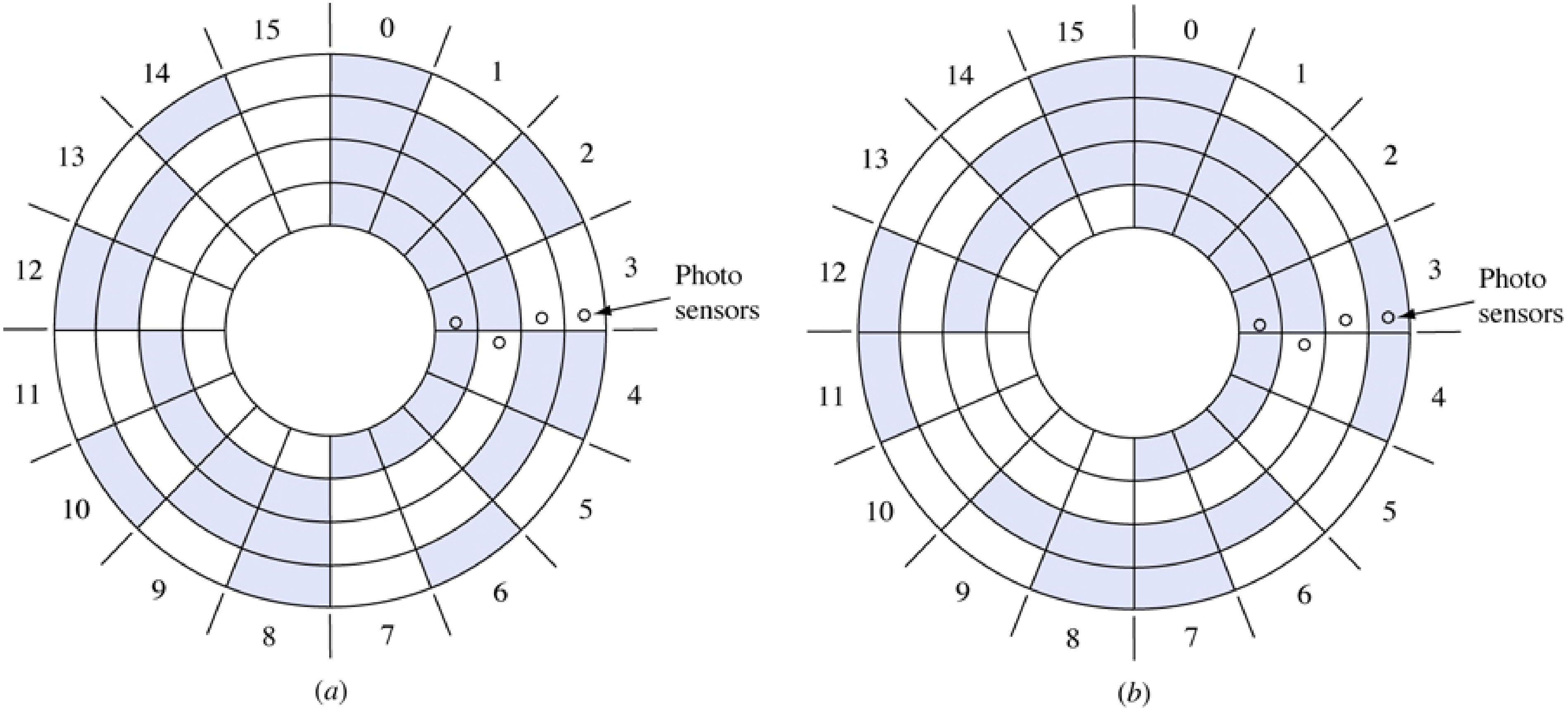


(*a*)



(*b*)

The effects of misaligned sensors on the encoders



Alphanumeric Codes

- ASCII code
- Unicode Standard

Error detection

- A parity bit can be used to detect single-bit errors.
- Additional parity bits can be used to detect multiple errors.

Examples of ASCII code with parity bit

characters	with even parity	with odd parity
.	.	.
.	.	.
.	.	.
A	10000010	10000011
B	10000100	10000101
C	10000111	10000110
D	10001000	10001001
.	.	.
.	.	.
.	.	.

Error correction

- Hamming code
- Use of check-sum digits

Hamming code

- Hamming code is an error-detection and error-correction binary code.
- A single-bit error can be automatically corrected if we can determine which bit is in error.
- A single-bit error can be detected by using a parity bit.
- Multiple parity bits can be used to pinpoint the bit in error.

Hamming code

Bit position	1	2	3	4	5	6	7	8	9	10	11	12
Use	P_1	P_2	D_3	P_4	D_5	D_6	D_7	P_8	D_9	D_{10}	D_{11}	D_{12}
Scope of P_1	√		√		√		√		√		√	
Scope of P_2		√	√			√	√			√	√	
Scope of P_4				√	√	√	√					√
Scope of P_8								√	√	√	√	√

Example

For example, to construct the Hamming code of 00101110

Use	P_1	P_2	D_3	P_4	D_5	D_6	D_7	P_8	D_9	D_{10}	D_{11}	D_{12}
			0		0	1	0		1	1	1	0
Scope of P_1	√		√		√		√		√		√	
Scope of P_2		√	√			√	√			√	√	
Scope of P_4				√	√	√	√					√
Scope of P_8								√	√	√	√	√
Bit position	1	2	3	4	5	6	7	8	9	10	11	12

Example (continued)

Choose 0 for P_1 (assuming the use of even parity)

Use	P_1	P_2	D_3	P_4	D_5	D_6	D_7	P_8	D_9	D_{10}	D_{11}	D_{12}
	0		0		0	1	0		1	1	1	0
Scope of P_1	√		√		√		√		√		√	
Scope of P_2		√	√			√	√			√	√	
Scope of P_4				√	√	√	√					√
Scope of P_8								√	√	√	√	√
Bit position	1	2	3	4	5	6	7	8	9	10	11	12

Example (continued)

Choose 1 for P_2

Use	P_1	P_2	D_3	P_4	D_5	D_6	D_7	P_8	D_9	D_{10}	D_{11}	D_{12}
	0	**1**	0		0	1	0		1	1	1	0
Scope of P_1	√		√		√		√		√		√	
Scope of P_2		√	√			√	√			√	√	
Scope of P_4				√	√	√	√					√
Scope of P_8								√	√	√	√	√
Bit position	1	2	3	4	5	6	7	8	9	10	11	12

Example (continued)

Choose 1 for P_4

Use	P_1	P_2	D_3	P_4	D_5	D_6	D_7	P_8	D_9	D_{10}	D_{11}	D_{12}
	0	**1**	0	**1**	0	1	0		1	1	1	0
Scope of P_1	√		√		√		√		√		√	
Scope of P_2		√	√			√	√			√	√	
Scope of P_4				√	√	√	√					√
Scope of P_8								√	√	√	√	√
Bit position	1	2	3	4	5	6	7	8	9	10	11	12

Example (continued)

Choose 1 for P_8

Use	P_1	P_2	D_3	P_4	D_5	D_6	D_7	P_8	D_9	D_{10}	D_{11}	D_{12}
	0	**1**	0	**1**	0	1	0	**1**	1	1	1	0
Scope of P_1	√		√		√		√		√		√	
Scope of P_2		√	√			√	√			√	√	
Scope of P_4				√	√	√	√					√
Scope of P_8								√	√	√	√	√
Bit position	1	2	3	4	5	6	7	8	9	10	11	12

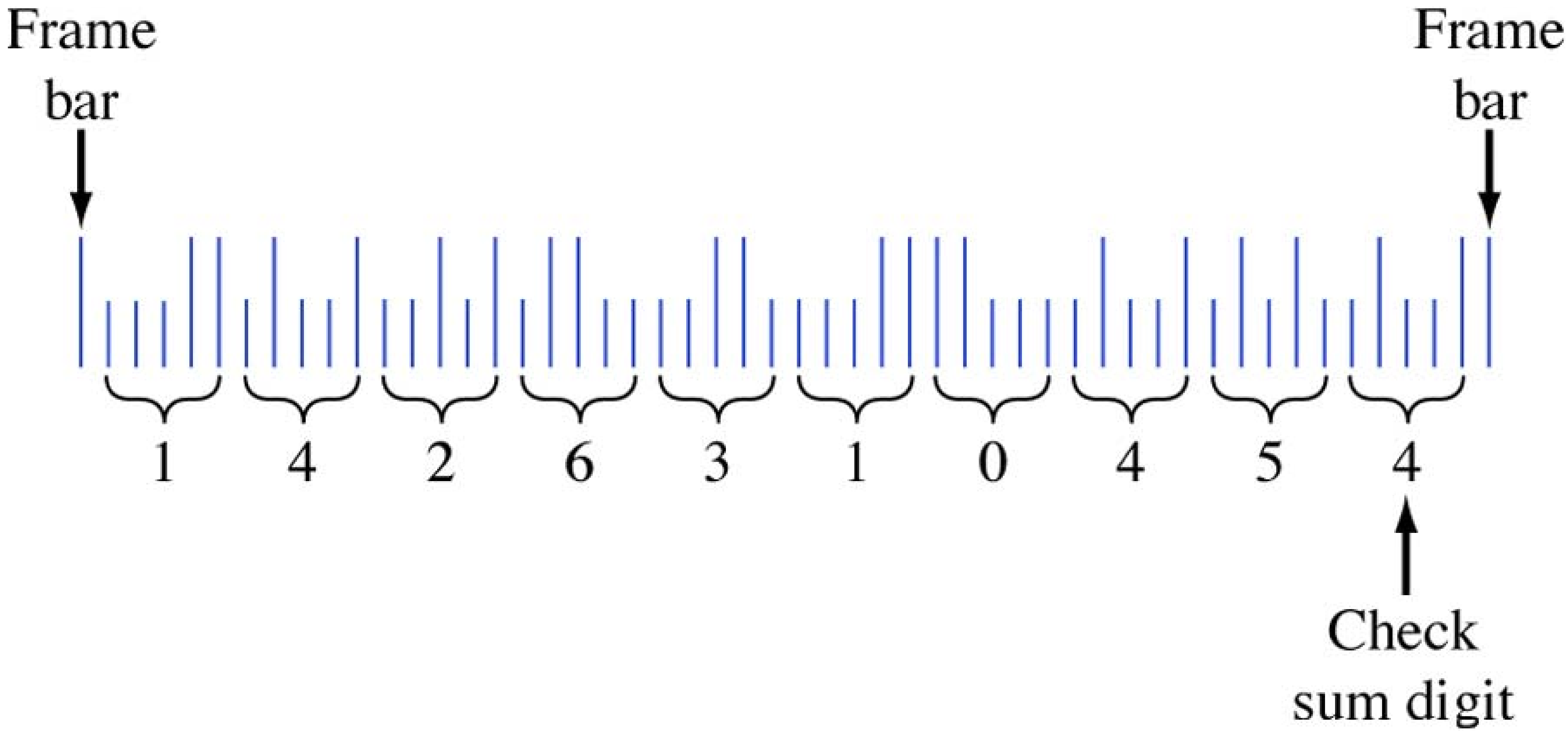
General properties of Hamming code

- It can be used for any code words with m information bits. It uses k parity bits such that $m \leq 2^k - k - 1$.
- By adding an additional parity bit to a Hamming code, we will be able to achieve single-error correction and double-error detection.

Check sum digit is inserted to satisfy the relation:

ZIP digit sum + check sum digit = 0 modulo 10

to make error correction possible. (Error detection is achieved by using the 2-out-of-5 code of individual digit)


Frame bar
Frame bar
1
4
2
6
3
1
0
4
5
4
Check sum digit