Vector Field Analysis
Other Features
Topological Features

- Flow recurrence and their connectivity

- Separation structure that classifies the particle advection
Vector Field Gradient Recall

• Consider a vector field
  \[ \frac{dx}{dt} = V(x) = \mathbf{f}(x, y, z) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \]

• Its gradient is
  \[ \nabla V = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{bmatrix} \]

It is also called the Jacobian matrix of the vector field. Many feature detection for flow data relies on Jacobian
Divergence and Curl

- **Divergence** - measures the magnitude of outward flux through a small volume around a point

\[
div V = \nabla \cdot V = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}
\]

\[
\nabla = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{bmatrix}
\]

- **Curl** - describes the infinitesimal rotation around a point

\[
curl V = \nabla \times V = \begin{bmatrix}
\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} & \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} & \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}
\end{bmatrix}
\]

\[
\nabla \cdot (\nabla \times V) = 0
\]

\[
\nabla \times (\nabla \phi) = 0
\]
Gauss Theorem

• Also known as divergence theorem, that relates the vectors on the boundary $\partial \mathcal{V} = \mathcal{A}$ of a region $\mathcal{V}$ to the divergence in the region

$$
\int_{\mathcal{V}} \text{div } \mathbf{V} d\mathcal{V} = \int_{\mathcal{A}} \mathbf{V} \cdot \mathbf{n} d\mathcal{A}
$$

$n$ being the outward normal of the boundary

• This leads to a physical interpretation of the divergence. Shrinking $\mathcal{V}$ to a point in the theorem yields that the divergence at a point may be treated as the material generated at that point
Stoke Theorem

- The rotation of vector field $V$ on a surface $\mathcal{A}$ is related to its boundary $\partial \mathcal{A} = \mathcal{L}$. It says that the curl on $\mathcal{A}$ equals the integrated field over $\mathcal{L}$.

$$\int_{\mathcal{A}} n \cdot \text{curl } V d\mathcal{A} = \oint_{\mathcal{L}} V \cdot dr$$

- This theorem is limited to two dimensional vector fields.
Another Useful Theorem about Curl

• In the book of Borisenko [BT79]
• Suppose $V = V' \times c$, an arbitrary but fixed vector, substituted into the divergence theorem. Using $\text{div}(V' \times c) = c \cdot \text{curl} V'$, one gets

$$\int_V \text{curl} V' \, dV = \int_{\mathcal{A}} n \times V' \, d\mathcal{A}$$

• Stoke’s theorem says that the flow around a region determines the curl.

• The second theorem says: Shrinking the volume $V$ to a point, the curl vector indicates the axis and magnitude of the rotation of that point.
2D Vector Field Recall

• Assume a 2D vector field
  \[ \frac{d\mathbf{x}}{dt} = V(x) = \mathbf{f}(x, y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} ax + by + c \\ dx + ey + f \end{pmatrix} \]

• Its Jacobian is
  \[ \nabla V = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \]

• Divergence is \( a + e \)
• Curl is \( b - d \)

Given a vector field defined on a discrete mesh, it is important to compute the coefficients \( a, b, c, d, e, f \) for later analysis.
Examples of Divergence and Curve of 2D Vector Fields

Divergence and curl of a vector field
Potential or Irrotational Fields

• A vector field $V$ is said to be a potential field if there exists a scalar field $\varphi$ with

$$V = \nabla \varphi$$

$\varphi$ is called the scalar potential of the vector field $V$

• A vector field $V$ living on a simply connected region is irrotational, i.e. $\text{curl } V = 0$ (i.e. curl-free), if and only if it is a potential field.

• It is worth noting that the potential defining the potential field is not unique, because

$$\nabla (U + c) = \nabla U + \nabla c = \nabla U + 0 = \nabla U$$
Solenoidal Fields

• Or divergence-free field
  \[ V = \text{curl} \Phi = \nabla \times \Phi \]

• Solenoidal fields stem from potentials too, but this time from vector potentials, \( \Phi \).

• These fields can describe incompressible fluid flow and are therefore as important as potential fields.

• A vector field \( V \) is solenoidal, i.e. \( V = \nabla \times \Phi \) with \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m \), if and only if the divergence of \( V \) vanishes.

• The vector potential here is not unique as well
  \[ \text{curl} (V + \nabla U) = \text{curl} V + \text{curl} \nabla U = \text{curl} V + 0 = \text{curl} V \]
Laplacian Fields

• A vector field $V$ which is both potential and solenoidal (i.e. both curl-free and divergence-free), is called a Laplacian field.

• In a simply connected region, a Laplacian field is the gradient of a scalar potential which satisfies Laplace differential equation $\Delta \varphi = 0$.

• Scalar function like $\varphi$ whose Laplacian vanishes, are called harmonic functions.
  – They are completely determined by their boundary values.
  – There exists one function satisfying Laplace’s equation for fixed boundary values.
Helmholtz Decomposition

\[ V = \nabla \phi + \nabla \times \Phi \]

- Curl (or rotation) free
- Divergence free

Hodge decomposition

\[ V = \nabla \phi + \nabla \times \Phi + \gamma \]

- Curl (or rotation) free
- Divergence free
- Harmonic
Helmholtz Decomposition Example

curl-free  neither  divergence-free
General Feature Classifications

• Points
  – Fixed points, vortex centers

• Lines
  – Features that occupy a set of points forming a line
  – 3D vortex cores, ridge lines, separation/attachment lines, cycles

• Surfaces
  – Features cover a set of points representing a surface
  – Shock wave, iso-surfaces, separation surfaces in 3D

• Volume
  – Features cover a non-zero region in 3D
  – Vortex region, 3D Morse sets, coherent structure
One important non-topological features in vector fields is vortex
Applications
Vortex Definition

• No rigorous and widely-accepted definition
• Capturing some swirling behavior

• Robinson 1991:
  – “A vortex exists when instantaneous *streamlines* mapped onto a plane normal to the vortex core exhibit a roughly circular or spiral pattern, when viewed from a reference frame moving with the center of the vortex core”

• Requires a priori detection
• Not always Galilean invariant: varying by adding constant vector fields
Different Definitions

• A vortex?

• [lugt’72]
  – A vortex is the rotating motion of a multitude of material particles around a common center
  – Vorticity is sufficiently strong – not enough to detect
Different Definitions

• A vortex?
• [Robinson’91]
  – A vortex exists when its streamlines, mapped onto a plane normal to its core, exhibit a circular or spiral pattern, under an appropriate reference frame
Different Definitions

• A vortex?
• [Portela’97]
  – A vortex is comprised of a central core region surrounded by swirling streamlines
Vortex Structures

• Two main classes of vortex structures
  – Region based methods: isosurfaces of scalar fields
  – Line based methods: extract vortex core lines
Region Based

• Threshold on **pressure**: 
  \[ p \leq p_{\text{thresh}} \]

• Idea: centripetal force induces pressure gradient
  – Very easy to implement and compute
  – Purely local criterion

• Problems:
  – Arbitrary threshold
  – Pressure can vary greatly along a vortex
Region Based

• Threshold on **vorticity magnitude**:
  \[ |\nabla \times V| \geq \omega_{thresh} \]

• Idea: strong infinitessimal rotation
  – Common in fluid dynamics community
  – Very easy to implement and compute, purely local

• Problems:
  – Arbitrary threshold
  – Vorticity often highest near boundaries
  – Vortices can have vanishing vorticity
Region Based

• Threshold on (normalized) helicity magnitude
  \[(\nabla \times \mathbf{V}) \cdot \mathbf{V} \geq h_{thresh}\]

• Idea: use vorticity but exclude shear flow
  – Still easy to implement and compute, purely local

• Problems:
  – Arbitrary threshold
  – Fails for curved shear layers
  – Vortices can have vanishing vorticity
Region Based

- $\lambda_2$-criterion

$$S := \frac{1}{2} (J + J^T)$$

Shear contribution of $J$

$$\Omega := \frac{1}{2} (J - J^T)$$

Rotational contribution of $J$

- Define as the largest eigenvalues of $S^2 + \Omega^2$
- Vortical motion where $\lambda_2 < 0$
  - Precise threshold, nearly automatic
  - Very widely used in CFD
  - Susceptible to high shear
  - Insufficient separation of close vortices
Region Based

- Q-criterion (Jeong, Hussain 1995)
- Positive 2\textsuperscript{nd} invariant of Jacobian
  \[ Q = \frac{1}{2} (\|\Omega\|^2 - \|S\|^2) \]
  
- Idea: \( Q > 0 \) implies local pressure smaller than surrounding pressure. Condition can be derived from characteristic polynomial of the Jacobian.

  - Common in CFD community
  - Can be physically derived from kinematic vorticity (Obrist, 1995)
  - Need good quality derivatives, can be hard to compute
Line Based

• Separation lines starting from focus saddle critical points [Globus/Levit 92]
Line Based

• Banks-Singer (1994):

  Idea: Assume a point on a vortex core is known.
  – Then, take a step in vorticity direction (predictor).
  – Project the new location to the pressure minimum perpendicular to the vorticity (corrector).
  – Break if correction is too far from prediction
Line Based

• Banks-Singer, continued

• Results in core lines that are roughly vorticity lines and pressure valleys.
  
  – Algorithmically tricky
  – Seeding point set can be large (e.g. local pressure minima)
  – Requires additional logic to identify unique lines
Line Based

• [Sujudi, Haimes 95]

• In 3D, in areas of 2 imaginary eigenvalues of the Jacobian matrix: *the only real eigenvalue is parallel to V*
  
  - In practice, standard method in CFD, has proven successful in a number of applications
  - Criterion is local per cell and readily parallelized
  - Resulting line segments are disconnected (Jacobian is assumed piecewise linear)
  - Numerical derivative computation can cause noisy results
  - Has problems with curved vortex core lines (sought-for pattern is straight)
Line Based

• Eigenvector method [Sujudi and Haimes 95]

\[ w(\mathbf{x}) = v(\mathbf{x}) - (v(\mathbf{x}) \cdot e(\mathbf{x}))e(\mathbf{x}) \]

\[ w(\mathbf{x}) = 0 \]

Reduced velocity

Although a point \( \mathbf{x} \) on the core structure is surrounded by spiraling integral curves, the flow vector at \( \mathbf{x} \) itself is solely governed by the non-swirling part of the flow.
Line Based

• Sahner et al. 2005

• Idea: construct a special vector field that allows to model ridge/valley-lines as integral curves (“feature flow field”).

• Authors applied it to Q-criterion and $\lambda_2$–criterion.
  – Works well in practice
  – Feature flow field requires high-order partial derivatives that are difficult to compute in certain data sets
  – Seed point set required (usually minimal points)
Line Based

- Sahner et al., results
Line Based

- The parallel vector operator
  [Roth and Peikert98]

- Given: 3D vector field $V$
- The curvature vector of $v$ is
  \[ c = \frac{v \times a}{|v|^3} \]
  where $a = \frac{Dv}{Dt}$ is the acceleration of

  Let $b = \frac{D^2v}{Dt^2}$

- Vortex core line: all locations in the domain where $b$ is parallel to $v$,

- Line structures
• Particular parallel vector approaches

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From M. Roth’s PhD thesis
Extend to Unsteady Flow

• Path lines: cores of swirling particle motion

[Weinkauf et al., Vis 2007]
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