Sub-Topics

• Compute bounding box
• Compute Euler Characteristic
• Estimate surface curvature
• Line description for conveying surface shape
• Extract skeletal representation of shapes
• Morse function and surface topology--Reeb graph
• Scalar field topology--Morse-Smale complex
Surface Topology – Reeb Graph
What is Topology?

- Topology studies the connectedness of spaces
- For us: how shapes/surfaces are connected
What is Topology

• The study of property of a shape that does not change under *deformation*
  
  – Rules of deformation
    • 1-1 and onto
    • Bicontinuous (continuous both ways)
    • Cannot tear, join, poke or seal holes
  
  – A is *homeomorphic* to B
Why is Topology Important?

• What is the **boundary** of an object?

• Are there **holes** in the object?

• Is the object **hollow**?

• If the object is transformed in some way, are the changes **continuous** or abrupt?

• Is the object **bounded**, or does it extend infinitely far?
Why is Topology Important?

• Inherent and basic properties of a shape
• We want to accurately represent and preserve these properties in different applications
  – Surface reconstruction
  – Morphing
  – Texturing
  – Simplification
  – Compression

Image source: http://www.utdallas.edu/~xxg061000/physicsmorphing.htm
Topology-Basic Concepts

• **n-manifold**
  
  – Set of points $M \subset \mathbb{R}^m$
  
  – Each point has a neighborhood homeomorphic to an open set of $\mathbb{R}^n$
  
  – An *n-manifold* is a topological space that “*locally looks like*” the Euclidean space $\mathbb{R}^n$
Topology-Basic Concepts

• **Holes/genus**
  - Genus of a surface is the maximal number of *nonintersecting simple closed curves* that can be cut on the surface without disconnecting it.

• **Boundaries**
Topology-Basic Concepts

- Euler’s characteristic function $\chi$
  $\chi(M) = V - E + F$
  - $V = \#$vertices, $E = \#$edges, $F = \#$faces
  - $\chi(M)$ is independent of the polygonization
  - Specifically, $\chi(M) = 2c - 2g - h$
    - What are $c$, $g$, and $h$?

\[ \chi = 1 \quad \chi = 0 \quad \chi = 2 \quad \chi = 0 \]
Topology-Basic Concepts

• Orientability
  – Any surface has a triangulation
  – Orient all triangles CW or CCW
  – Orientability: any two triangles sharing an edge have opposite directions on that edge.
  – Can distinguish in and out of the surface
Morse Theory

• Investigates the topology of a surface by looking at *critical points* of a function on that surface. \( \nabla f(p) = \left( \frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p) \right) = 0 \)

• \( f: M \rightarrow R \)
Morse Functions

• A function $f$ is a Morse function if
  – $f$ is smooth
  – All critical points are isolated
  – All critical points are non-degenerate $\det(\text{Hessian}(p)) \neq 0$

$$\text{Hessian } f(p) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\
\frac{\partial^2 f}{\partial y \partial x}(p) & \frac{\partial^2 f}{\partial y^2}(p)
\end{pmatrix}$$

– A non-Morse function can be made Morse by adding small but random noise
Notion of Critical Points

- Points where $\nabla f$ vanishes
  - Minima, maxima, and saddles
  - Correspond to places where the topology of the function changes
  - The function behavior can be illustrated by isolines/level sets on the domain

- Level set of a given value $i$
  \[ f^{-1}(i) = \{ p \in M \mid f(p) = i \} \]
Example – dunking a doughnut

• $f(p) = z$ (height function)

Shape analysis is a special case of scalar field analysis
Example – dunking a doughnut
Example – dunking a doughnut
Example – dunking a doughnut
Example – dunking a doughnut
Example – dunking a doughnut
Notion of Critical Point

• Points where $\nabla f$ vanishes
  – Minima, maxim, and saddles
  – Topological changes
  – Piecewise linear interpolation
  – Barycentric coordinates on triangles
  – Only exist at vertices
Notion of Critical Point

• Combinatorial identification
  – One-ring neighborhood (or a star) of a vertex $\nu$
    • Triangles and edges that adjacent to $\nu$
Notion of Critical Point

• Combinatorial identification
  – One-ring neighborhood (or a star) of a vertex \( \nu \)
    • Triangles and edges that adjacent to \( \nu \)
  – Link of a vertex
    • Edges of the star that do not adjacent to \( \nu \)
Notion of Critical Point

• Combinatorial identification
  – Lower link of $\nu$
    • Edges of the link of $\nu$ whose function values are strictly smaller than $f(\nu)$
Notion of Critical Point

• Combinatorial identification
  – Lower link of \( v \)
    • Edges of the link of \( v \) whose function values are strictly smaller than \( f(v) \)
  – Upper link of \( v \)
    • Edges of the link of \( v \) whose function values are strictly larger than \( f(v) \)
Notion of Critical Point

- Combinatorial identification
  - Minima
    - Empty lower link
Notion of Critical Point

- Combinatorial identification
  - Minima
    - Empty lower link
  - Maxima
    - Empty upper link
Notion of Critical Point

- Combinatorial identification
  - Minima
    - Empty lower link
  - Maxima
    - Empty upper link
  - Regular point
    - Lower and upper links both simply connected
Notion of Critical Point

• Combinatorial identification
  – Everything else
    • Saddle
      – Multiple connected lower and upper links
  – Works in arbitrary dimension if replace triangles, edges, and vertices with simplices
Notion of Critical Point

- Combinatorial identification
  - Everything else
    - Saddle
      - Multiple connected lower and upper links
  - Works in arbitrary dimension if replace triangles, edges, and vertices with \textit{simplices}
  - \textit{Value of a critical point} – \textit{critical value}
How Does it Work?

Morse function is defined as the height function (i.e. z coordinates)
Level sets obtaining by sweeping along Z direction
Review of Level Sets

• Given a domain $M$ of dimension $d$
  – Level set of a given value $i$, $f^{-1}(i) = \{p \in M \mid f(p) = i\}$

• If $i$ is not a critical value
  – $f^{-1}(i)$ is a $(d - 1)$ manifold

• If $M$ is closed, $f^{-1}(i)$ is closed.
  Otherwise it may be open
Level Set Extraction

• Given a triangular mesh $X$ and a scalar function $f$, the level set of a given value $s^*$ is compute as follows:
  
  – First, determine the intersections of the level set $f^{-1}(s^*)$ with the edges of $X$

  – Second, traverse through all triangles and connect the obtained intersections.
Which Edges Intersect with $f^{-1}(s^*)$?

Increasing function values

- $s_0, s_1 > s^*$
- $s_0, s_1 < s^*$
- $s_1 = s^*$
- $s_0 = s^*$
- $s_0 < s^* < s_1$
Determine Intersections with Edges

Does $f^{-1}(s^*)$ cross any edges of this square?

Linearly interpolating the scalar value from node 0 to node 1 gives:

$$S = (1 - t)S_0 + tS_1 = S_0 + t(S_1 - S_0) \quad \text{where } 0 \leq t \leq 1.$$

Setting this interpolated $S$ equal to $S^*$ and solving for $t$ gives:

$$t^* = \frac{S^* - S_0}{S_1 - S_0}$$
If $0. \leq t^* \leq 1.$, then $S^*$ crosses this edge. You can compute where $S^*$ crosses the edge by using the same linear interpolation equation you used to compute $S^*$. You will need that for later visualization.

\[
x^* = (1 - t^*)x_0 + t^*x_1 \\
y^* = (1 - t^*)y_0 + t^*y_1
\]
Connect Intersections

No intersection
Connect Intersections

No intersection

Two intersections
Connect Intersections

No intersection

Two intersections

One or three intersections
Connect Intersections

No intersection

Two intersections

One or three intersections

Invalid
Reeb Graph

• Given a Morse function
  \[ f: M \rightarrow R \]
Reeb Graph

- Given a Morse function

\[ f: M \rightarrow R \]
Reeb Graph

• Given a Morse function
  \[ f : M \to R \]
Reeb Graph

- Given a Morse function
  \[ f : M \rightarrow R \]
Reeb Graph

- Given a Morse function
  \[ f: M \to \mathbb{R} \]
Reeb Graph

• Given a Morse function
  \( f : M \to R \)
Reeb Graph

• Given a Morse function
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Reeb Graph

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• Given a Morse function
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Reeb Graph

- Given a Morse function
  \[ f: M \to R \]
Reeb Graph

• Given a Morse function
  \[ f: M \to R \]

  • Reeb graph \( R(f) \)
  • Contour retraction of \( M \) under \( f \)
Reeb Graph

- Vertices of the graph are critical points
- Arcs of the graph are connected components (cylinders in domain) of the level sets of $f$, contracted to points
Reeb Graph Properties

- Continuous 1-dimensional simplicial complex
- Extrema: valence 1
- Saddles in 2D: valence 3 or 4 (boundary)
- Saddles in nD: valence 2 or 3

In practice:
  - Arcs: collection of vertices
  - Simply connected domains: Carr00
  - Otherwise: Pascucci07, Tierny09, Parsa12
Reeb graphs and genus

- The number of loops in the Reeb graph is equal to the surface genus
- To count the loops, simplify the graph by contracting degree-1 vertices and removing degree-2 vertices
Compute Reeb Graph Using Cylinder Maps

• Cylinders in domain map to arcs in Reeb graph
• Two step algorithm
  – Step I: Locate critical points
    • Sort the critical points based on their scalar value
    • compute the critical level set corresponding to each critical point

[Natarajan 2011]
Step II - Connect Critical Points

- Cylinder represents evolution of one level set component
- Trace all level set components within a cylinder in an iteration

[Natarajan 2011]
Step II - Connect Critical Points

- Is-graph: stores adjacencies between triangles
- For each component
  - Start with a triangle in the component attaching to the upper link of a critical point
  - Traverse the edges until a triangle in a critical level set is reached
  - Insert corresponding arc in the Reeb graph

[Natarajan 2011]
Reeb Graph Visualization

• Embedded layout

[Natarajan 2011]
Some More Reeb Graph Examples
Discretized Reeb Graph

- Take the critical points and “samples” in between
- Robust because we know that nothing happens between consecutive critical points
Reeb graphs for Shape Matching

• Reeb graph encodes the behavior of a Morse function on the shape
• Also tells us about the topology of the shape
• Take a meaningful function and use its Reeb graph to compare between shapes!
Choose the right Morse function

- The height function $f(p) = z$ is not good enough – not rotational invariant
- Not always a Morse function
Average geodesic distance

• The idea of [Hilaga et al. 01]: use geodesic distance for the Morse function!

\[ g(\mathbf{p}) = \int_M \text{geodist}(\mathbf{p}, \mathbf{q}) dS \]

\[ f(\mathbf{p}) = \frac{g(\mathbf{p}) - \min_{\mathbf{q} \in M} g(\mathbf{q})}{\max_{\mathbf{q} \in M} g(\mathbf{q})} \]
Multi-res Reeb graphs

• Hilaga et al. use multiresolutional Reeb graphs to compare between shapes
• Multiresolution hierarchy – by gradual contraction of vertices
Reeb Graph Applications

Flexible isosurfaces
[Carr et al., IEEE VIS 2004]

Topology controlled volume rendering
[Weber et al., TVCG 2007]
Reeb Graph Applications

Seed sets for isosurfaces
[van Kreveld et al., SoCG 1997]

Surface parameterization
[Zhang et al., TOG 2005]

Topological simplification and cleaning
[Wood et al., TOG 2004; Pascucci et al., SIGGRAPH 2007]
Reeb Graph Applications

Mesh segmentation [Zhang et al. 2003]
Reeb Graph Applications

Topology-based shape matching
[Hilaga et al., SIGGRAPH 2001]
Additional Reading for Reeb Graph

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