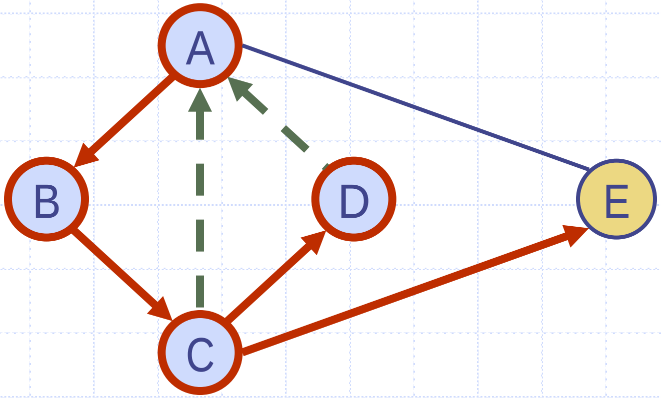
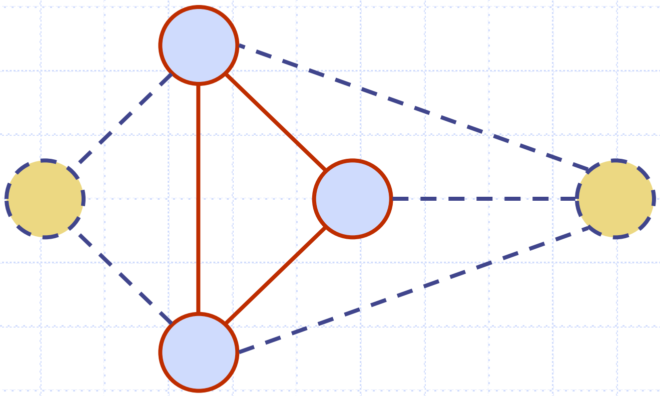
Depth-First Search

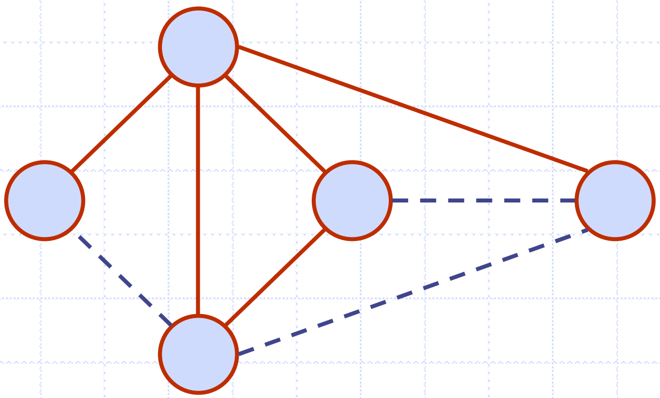


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



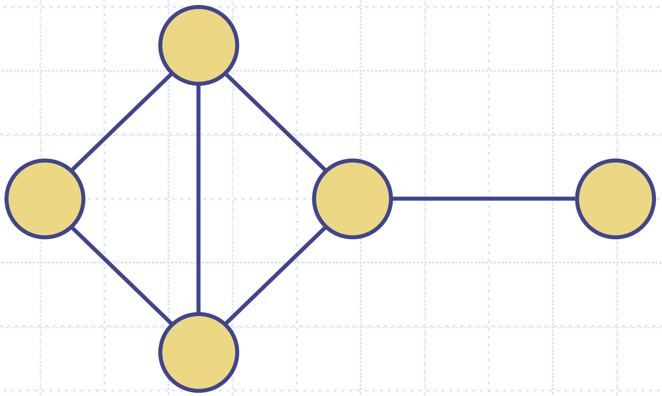
Subgraph



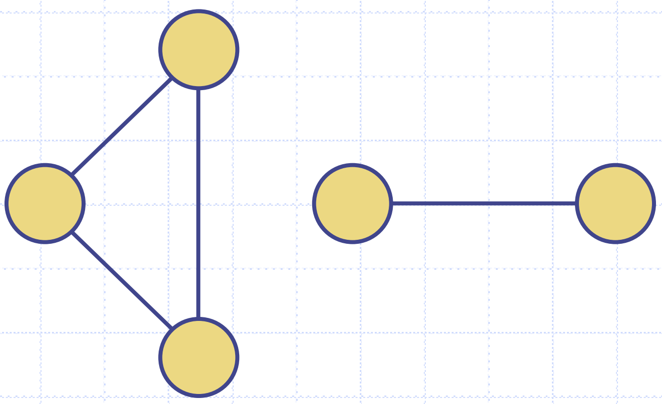
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph

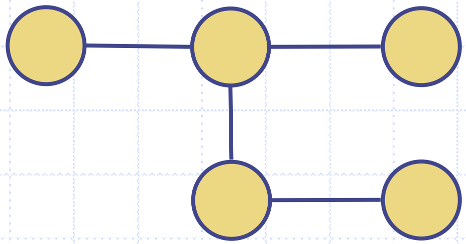


Non connected graph with two connected components

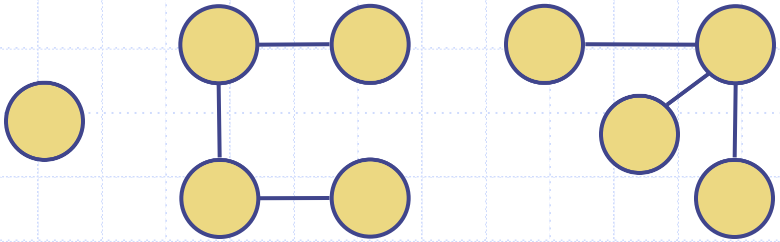
Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



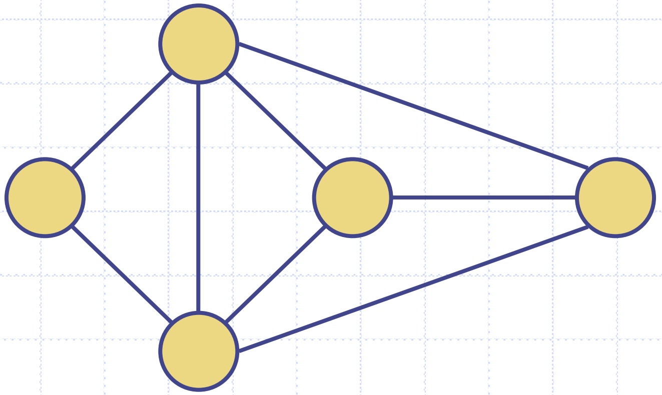
Tree



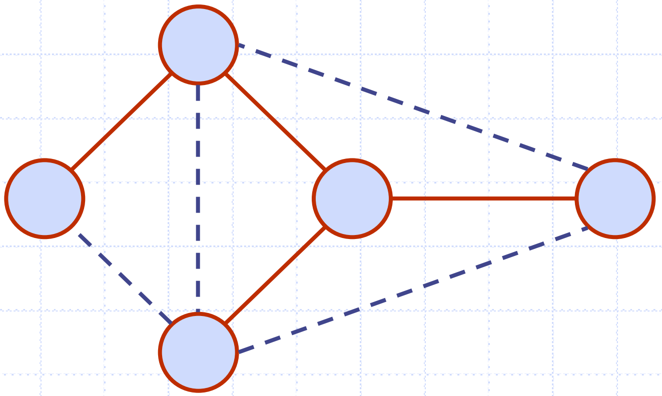
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

DFS Algorithm

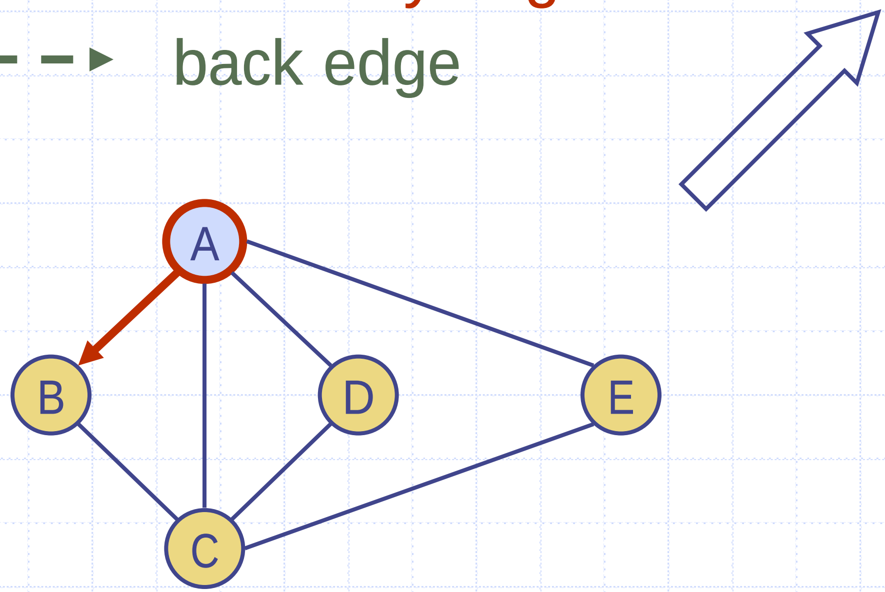
- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

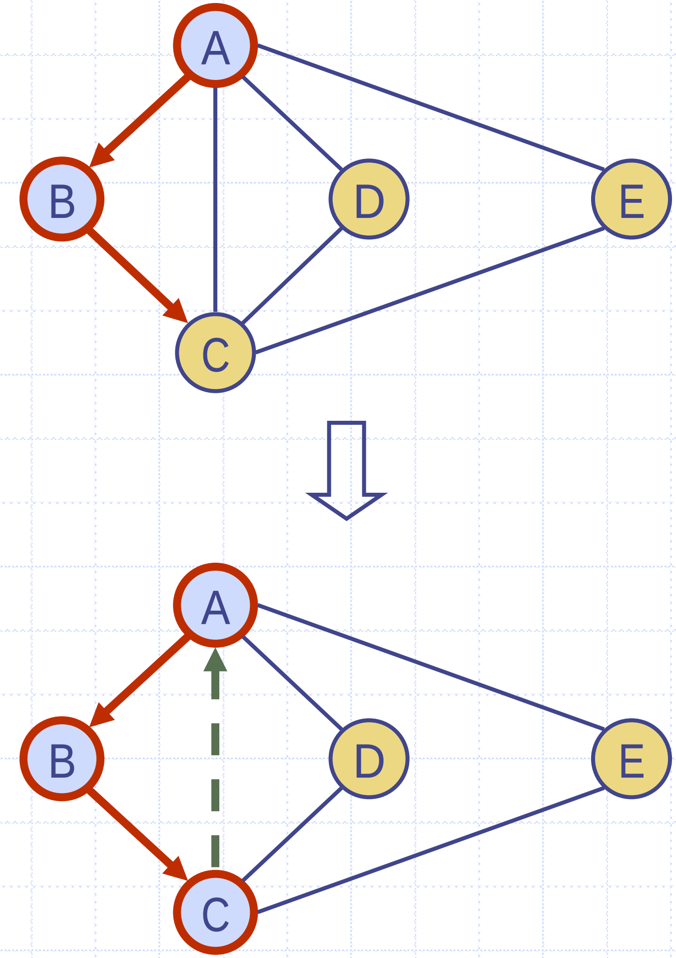
Algorithm *DFS*(*G*)
    Input graph *G*
    Output labeling of the edges of *G* as discovery edges and back edges
  for all $u \in G.vertices()$
    $u.setLabel(UNEXPLORED)$
  for all $e \in G.edges()$
    $e.setLabel(UNEXPLORED)$
  for all $v \in G.vertices()$
    if $v.getLabel() = UNEXPLORED$
      $DFS(G, v)$

Algorithm *DFS*(*G*, *v*)
  Input graph *G* and a start vertex *v* of *G*
  Output labeling of the edges of *G* in the connected component of *v* as discovery edges and back edges
  $v.setLabel(VISITED)$
  for all $e \in G.incidentEdges(v)$
    if $e.getLabel() = UNEXPLORED$
      $w \leftarrow e.opposite(v)$
      if $w.getLabel() = UNEXPLORED$
        $e.setLabel(DISCOVERY)$
        $DFS(G, w)$
      else
        $e.setLabel(BACK)$

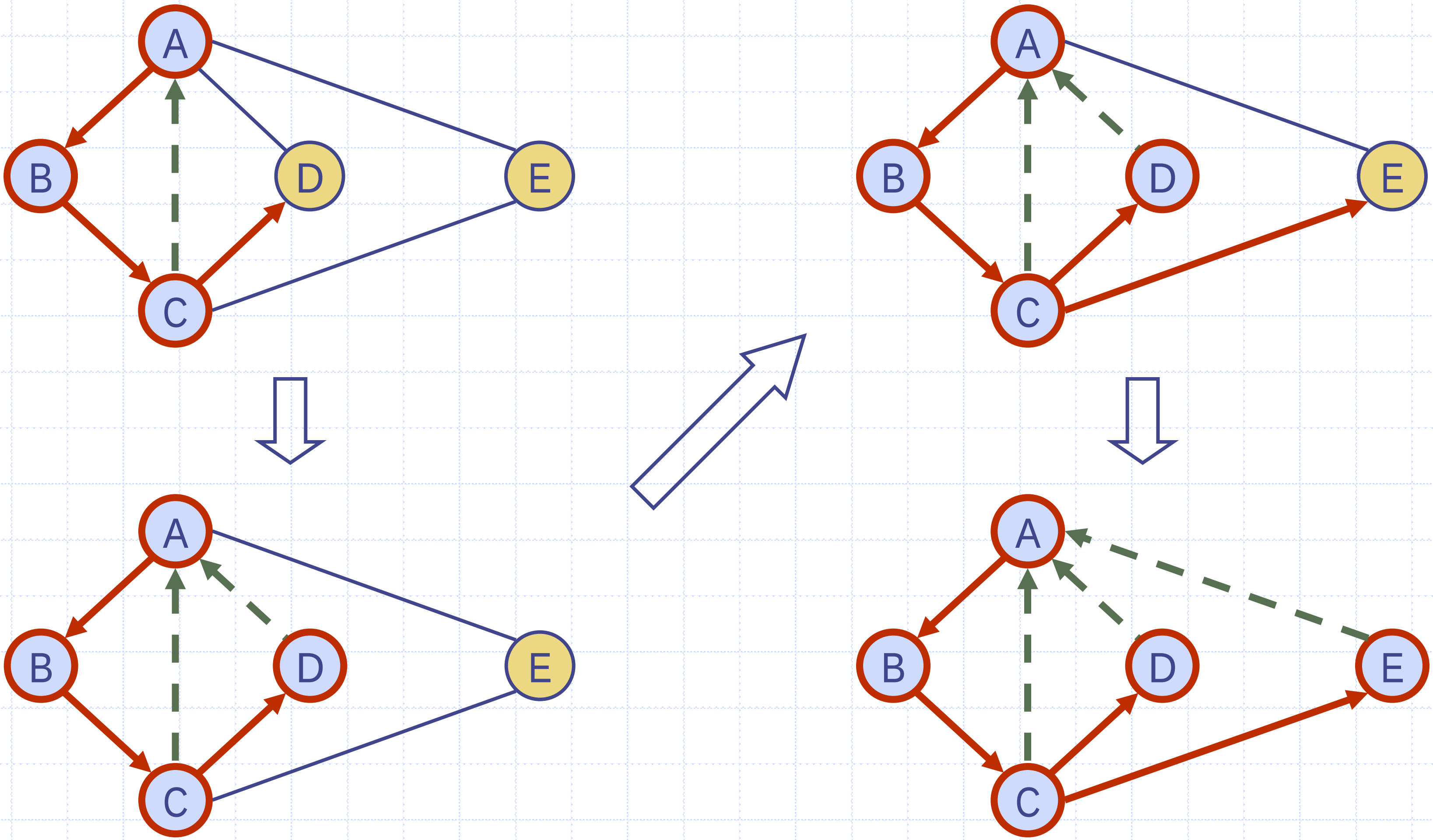
Example

- unexplored vertex
- visited vertex
- unexplored edge
- discovery edge
- back edge

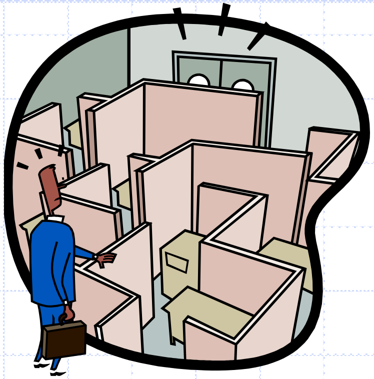




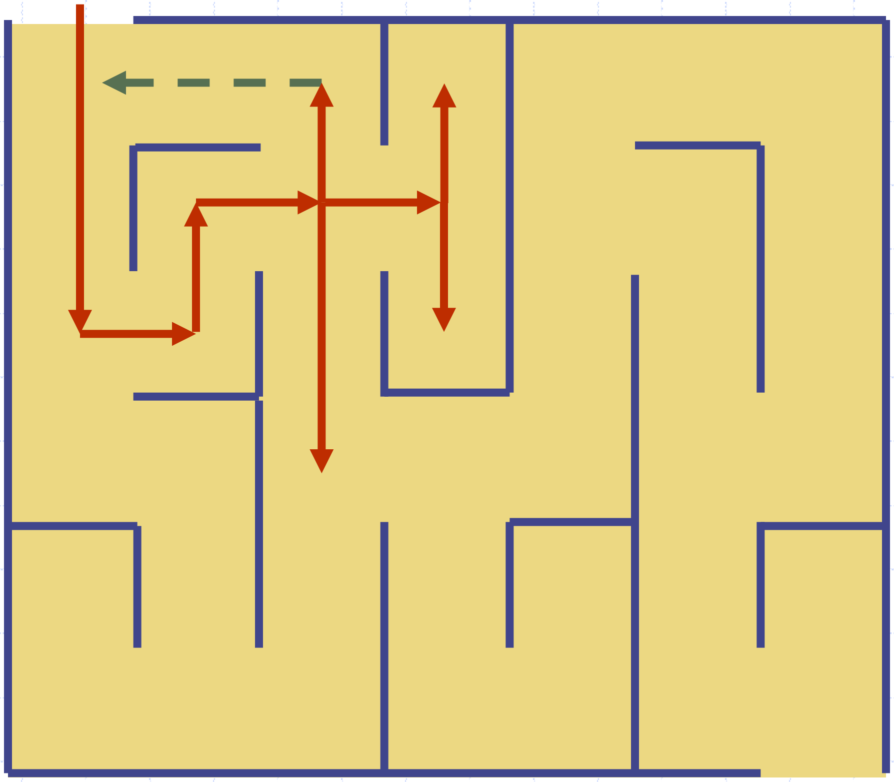
Example (cont.)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



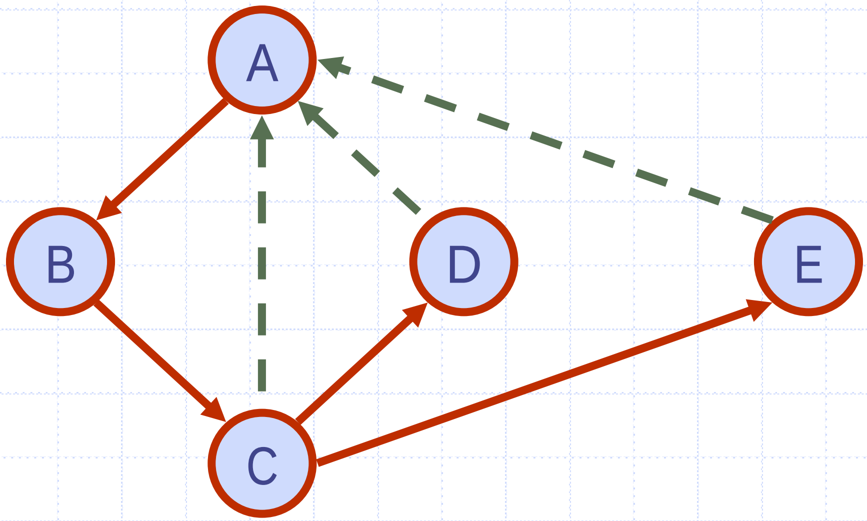
Properties of DFS

Property 1

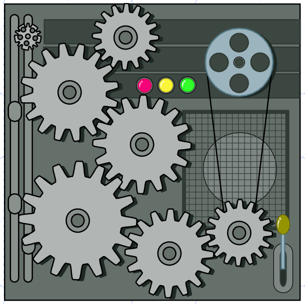
$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v



Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  v.setLabel(VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ v.incidentEdges()
    if e.getLabel() = UNEXPLORED
      w ← e.opposite(v)
      if w.getLabel() = UNEXPLORED
        e.setLabel(DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        e.setLabel(BACK)
  S.pop(v)
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v, z)
    v.setLabel(VISITED)
    S.push(v)
    for all  e ∈ v.incidentEdges()
        if  e.getLabel() = UNEXPLORED
            w ← e.opposite(v)
            S.push(e)
            if w.getLabel() = UNEXPLORED
                e.setLabel(DISCOVERY)
                pathDFS(G, w, z)
                S.pop(e)
            else
                T ← new empty stack
                repeat
                    o ← S.pop()
                    T.push(o)
                until o = w
                return T.elements()
    S.pop(v)
```