# Priority Queues

# Priority Queue ADT

- A priority queue stores a collection of entries
- Typically, an entry is a pair (key, value), where the key indicates the priority
- Main methods of the Priority Queue ADT
  - insert(e)
    inserts an entry e
    removeMin()
    removes the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), empty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

# Total Order Relations

- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct entries in a priority queue can have the same key
- Mathematical concept of total order relation $\leq$
  - Reflexive property: $x \leq x$
  - Antisymmetric property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - Transitive property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

# Comparator ADT

- Implements the boolean function isLess(p,q), which tests whether p < q
- Can derive other relations from this:
  - (p == q) is equivalent to
  - (!isLess(p, q) && !isLess(q, p))
- Can implement in C++ by overloading "()"

Two ways to compare 2D points:

```
class LeftRight { // left-right comparator
public:
    bool operator()(const Point2D& p,
            const Point2D& q) const
    { return p.getX() < q.getX(); }
};
class BottomTop { // bottom-top
public:
    bool operator()(const Point2D& p,
    const Point2D& q) const
    { return p.getY() < q.getY(); }
};
```

# Priority Queue Sorting

- We can use a priority queue to sort a set of comparable elements
  1. Insert the elements one by one with a series of insert operations
  2. Remove the elements in sorted order with a series of removeMin operations
- The running time of this sorting method depends on the priority queue implementation

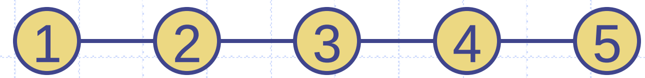
**Algorithm** *PQ-Sort*($S$, $C$)
    **Input** sequence $S$, comparator $C$ for the elements of $S$
    **Output** sequence $S$ sorted in increasing order according to $C$
    $P \leftarrow$ priority queue with comparator $C$
    **while** $\neg S.empty$ ()
        $e \leftarrow S.front()$; $S.eraseFront()$
        $P.insert\ (e, \emptyset)$
    **while** $\neg P.empty()$
        $e \leftarrow P.removeMin()$
        $S.insertBack(e)$

# Sequence-based Priority Queue

- Implementation with an unsorted list


- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list




- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

# Selection-Sort

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence
- Running time of Selection-sort:
  1. Inserting the elements into the priority queue with $n$ insert operations takes $O(n)$ time
  2. Removing the elements in sorted order from the priority queue with $n$ removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

# Selection-Sort Example

| | Sequence S | Priority Queue P |
|---|---|---|
| Input: | (7,4,8,2,5,3,9) | () |
| Phase 1 | | |
| (a) | (4,8,2,5,3,9) | (7) |
| (b) | (8,2,5,3,9) | (7,4) |
| .. | .. | .. |
| (g) | () | (7,4,8,2,5,3,9) |
| Phase 2 | | |
| (a) | (2) | (7,4,8,5,3,9) |
| (b) | (2,3) | (7,4,8,5,9) |
| (c) | (2,3,4) | (7,8,5,9) |
| (d) | (2,3,4,5) | (7,8,9) |
| (e) | (2,3,4,5,7) | (8,9) |
| (f) | (2,3,4,5,7,8) | (9) |
| (g) | (2,3,4,5,7,8,9) | () |

# Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence
- Running time of Insertion-sort:
  1. Inserting the elements into the priority queue with $n$ insert operations takes time proportional to
     $$1 + 2 + \ldots + n$$
  2. Removing the elements in sorted order from the priority queue with a series of $n$ removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

# Insertion-Sort Example

| | Sequence S | Priority queue P |
|---|---|---|
| Input: | (7,4,8,2,5,3,9) | () |
| | | |
| Phase 1 | | |
| (a) | (4,8,2,5,3,9) | (7) |
| (b) | (8,2,5,3,9) | (4,7) |
| (c) | (2,5,3,9) | (4,7,8) |
| (d) | (5,3,9) | (2,4,7,8) |
| (e) | (3,9) | (2,4,5,7,8) |
| (f) | (9) | (2,3,4,5,7,8) |
| (g) | () | (2,3,4,5,7,8,9) |
| | | |
| Phase 2 | | |
| (a) | (2) | (3,4,5,7,8,9) |
| (b) | (2,3) | (4,5,7,8,9) |
| .. | .. | .. |
| (g) | (2,3,4,5,7,8,9) | () |

# In-place Insertion-Sort

- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place
- A portion of the input sequence itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the sequence
  - We can use swaps instead of modifying the sequence