Chapter 4

Simplification of Boolean Expressions

Factors to be considered

in evaluating the merit of a network include

- Cost
- Reliability
- Propagation delay: the times it takes the network to respond to changes at its inputs

Cost measure

- The cost of a network can be measured in many different ways.
- In this course, unless otherwise specified, the cost of a network is defined to be the total number of gates plus the total number of gate inputs.
- Availability of inputs in complemented and uncomplemented form is assumed.

The cost of a Boolean expression

- is defined to be the cost of the corresponding network.

Relations among Boolean expressions

- Boolean expression f_1 *implies* Boolean expression f_2 if any assignment of values to the variables involved makes $f_1 = 1$, it also makes $f_2 = 1$. For example, $f_1 = x'z+y'z$ implies $f_2 = x'y + y'z$.
- A term t_1 subsumes term t_2 if any literal that occurs in t_2 also occurs in t_1. For example, $wx'yz$ subsumes $x'z$, and $x + y$ subsumes x.

Implicants

- A product term is said to be an implicant of a Boolean function if it implies the function.
- For example, if a function is expressed in sum of products, then any product term therein is an implicant. If $f(x, y) = x'y + xy'$ then $x'y$ is an implicant. So is xy'.

Prime implicants

- An implicant of a function is said to be a prime implicant if the implicant does not subsume any other implicant with fewer literals of that function.
- For example, consider $f(x, y, z) = x'y + z$. Both $x'y$ and xz are implicants of f. But while $x'y$ is a prime implicant of f, xz is not because it subsumes z.

Irredundant disjunctive normal formula

An irredundant disjunctive normal formula (IDNF) is a Boolean expression in sum-of-product form such that (1) every product term involved is a prime implicant, and (2) no product term may be eliminated without changing the definition of that function. For example, $x'y + z$ is an IDNF but not $x'y + z + x'z$ or $x'y$.

The minimization problem

- The the minimization problem to be discussed in the following is to find, for a given Boolean expression, an equivalent one that has the minimum cost, and that satisfies any other constraints imposed.

Simplification methods

- A graphic method that can handle Boolean expressions up to 6 variables - Karnaugh maps
- A tabular method that has no limit on the number of variable and can be implemented on a computer - Quine-McCluskey method

Graphic method

- It is a simplification method that makes use of the following relations:

  $x + x' = 1$

  $y \bullet 1 = 1 \bullet y = y$

- It facilitates recognition of applicability of these relations by describing a Boolean function in a graphic form (Karnaugh map).

Basic idea

- Two product terms of a Boolean function can be combined and simplified if they have a distance of 1.
- The distance between two product terms is defined as the number of literals that occur differently (i.e., one is complemented while the other is not) in these terms.

Karnaugh maps

- The Karnaugh map of a function consists of a number of cells (squares) that is equal to number of its minterms.
- Each cell is associated with a minterm in such a way that, if two cells are immediately adjacent to each other, their corresponding minterms have a distance of 1.

Karnaugh maps (continued)

- The entry to a cell is equal to the value of the associated minterm.
- Possibilities of simplification is signified by the presence of 1's occupying adjacent cells.
- A group of 2^n cells can be combined to form a simpler term.

Karnaugh maps and implicants

A Karnaugh map of a Boolean function allows us to visualize its implicants, prime implicants, and different ways to describe it as a irredundant disjunctive normal formula.

A different view

Conceptually, it may be useful to think of a Karnaugh map as a different form of the truth table. Each row of the truth table is embedded in a cell in in the map in such a way that the possibility of simplification becomes obvious.

Karnaugh map of function of one variable.

x	$f(x)$
0	$f(0)$
1	$f(1)$

(a)

x: 0	1
$f(0)$	$f(1)$

(b)

Map of a function of 2 variables

x	y	$f(x,y)$
0	0	$f(0,0)$
0	1	$f(0,1)$
1	0	$f(1,0)$
1	1	$f(1,1)$

(*a*)

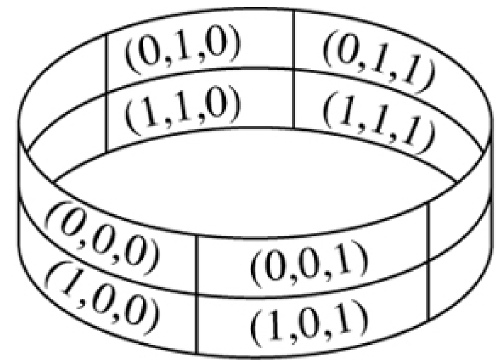
x \ y	0	1
0	$f(0,0)$	$f(0,1)$
1	$f(1,0)$	$f(1,1)$

(*b*)

Map of a function of 3 variables

x	y	z	$f(x,y,z)$
0	0	0	$f(0,0,0)$
0	0	1	$f(0,0,1)$
0	1	0	$f(0,1,0)$
0	1	1	$f(0,1,1)$
1	0	0	$f(1,0,0)$
1	0	1	$f(1,0,1)$
1	1	0	$f(1,1,0)$
1	1	1	$f(1,1,1)$

(*a*)



(*b*)

x \ yz	00	01	11	10
0	$f(0,0,0)$	$f(0,0,1)$	$f(0,1,1)$	$f(0,1,0)$
1	$f(1,0,0)$	$f(1,0,1)$	$f(1,1,1)$	$f(1,1,0)$

(*c*)

An example map

x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

(a)

x \ yz	00	01	11	10
0	1	0	0	1
1	1	1	0	0

(b)

Map of a function of 4 variables

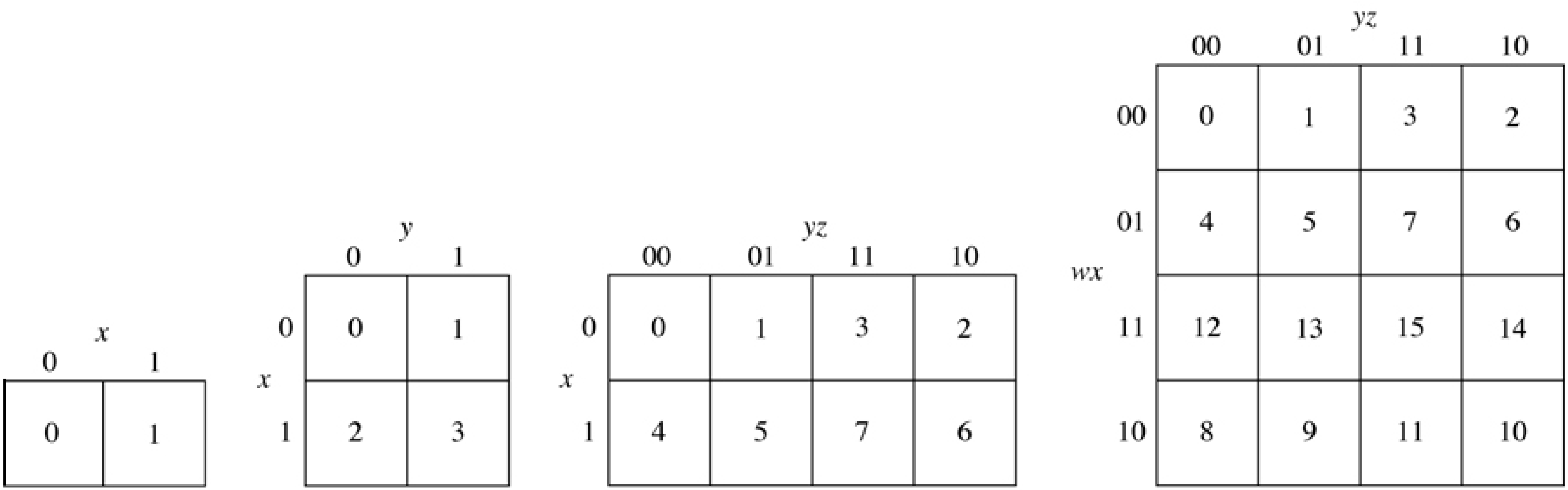
w	x	y	z	$f(w,x,y,z)$
0	0	0	0	$f(0,0,0,0)$
0	0	0	1	$f(0,0,0,1)$
0	0	1	0	$f(0,0,1,0)$
0	0	1	1	$f(0,0,1,1)$
0	1	0	0	$f(0,1,0,0)$
0	1	0	1	$f(0,1,0,1)$
0	1	1	0	$f(0,1,1,0)$
0	1	1	1	$f(0,1,1,1)$
1	0	0	0	$f(1,0,0,0)$
1	0	0	1	$f(1,0,0,1)$
1	0	1	0	$f(1,0,1,0)$
1	0	1	1	$f(1,0,1,1)$
1	1	0	0	$f(1,1,0,0)$
1	1	0	1	$f(1,1,0,1)$
1	1	1	0	$f(1,1,1,0)$
1	1	1	1	$f(1,1,1,1)$

(*a*)

wx \ yz	00	01	11	10
00	$f(0,0,0,0)$	$f(0,0,0,1)$	$f(0,0,1,1)$	$f(0,0,1,0)$
01	$f(0,1,0,0)$	$f(0,1,0,1)$	$f(0,1,1,1)$	$f(0,1,1,0)$
11	$f(1,1,0,0)$	$f(1,1,0,1)$	$f(1,1,1,1)$	$f(1,1,1,0)$
10	$f(1,0,0,0)$	$f(1,0,0,1)$	$f(1,0,1,1)$	$f(1,0,1,0)$

(*b*)

Maps with cells designated by decimal numbers



(a) (b) (c) (d)

A variant of Karnaugh map

- There are many different ways to construct Karnaugh maps.
- The ones shown next make the simplification process less error prone.

Map for a function of two variables

	y'	y
x'	f(0, 0)	f(0, 1)
x	f(1, 0)	f(1, 1)

Map of function of three variables

	y'z'	y'z	yz	yz'
x'	f(0,0,0) 0	f(0,0,1) 1	f(0,1,1) 3	f(0,1,0) 2
x	f(1,0,0) 4	f(1,0,1) 5	f(1,1,1) 7	f(1,1,0) 6

Map of function of four variables
f(w, x, y, z)

	y'z'	y'z	yz	yz'
w'x'	0	1	3	2
w'x	4	5	7	6
wx	12	13	15	14
wx'	8	9	11	10

Map of function of five variables
f(v, w, x, y, z)

	v'				v			
	y'z'	y'z	yz	yz'	yz'	yz	y'z	y'z'
w'x'	0	1	3	2	18	19	17	16
w'x	4	5	7	6	22	23	21	20
wx	12	13	15	14	30	31	29	28
wx'	8	9	11	10	26	27	25	24

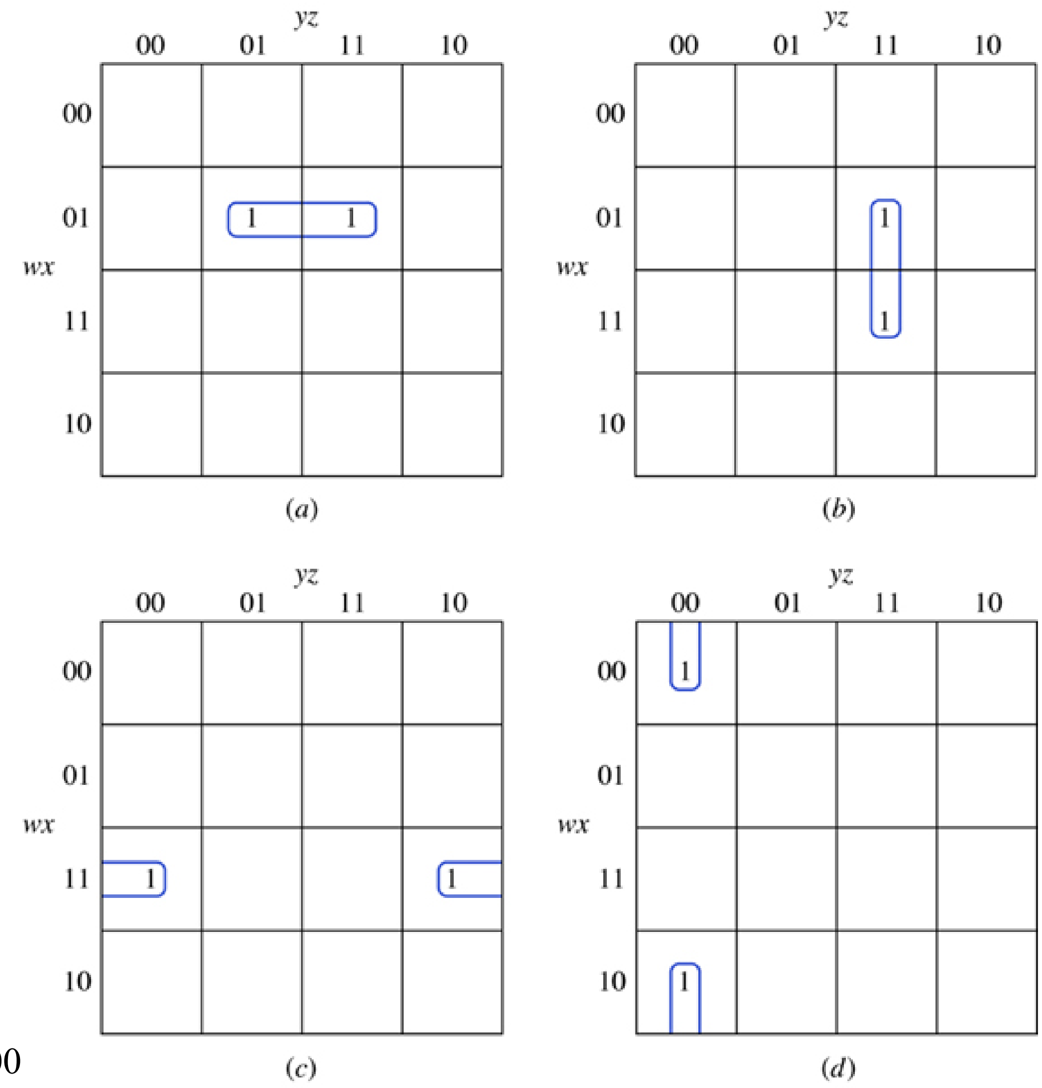
Map of function of six variables f(u, v, w, x, y, z)

		v'				v			
		y'z'	y'z	yz	yz'	yz'	yz	y'z	y'z'
u'	w'x'	0	1	3	2	18	19	17	16
	w'x	4	5	7	6	22	23	21	20
	wx	12	13	15	14	30	31	29	28
	wx'	8	9	11	10	26	27	25	24
u	wx'	40	41	43	42	58	59	57	56
	wx	44	45	47	46	62	63	61	60
	w'x	36	37	39	38	54	55	53	52
	w'x'	32	33	36	34	50	51	49	48

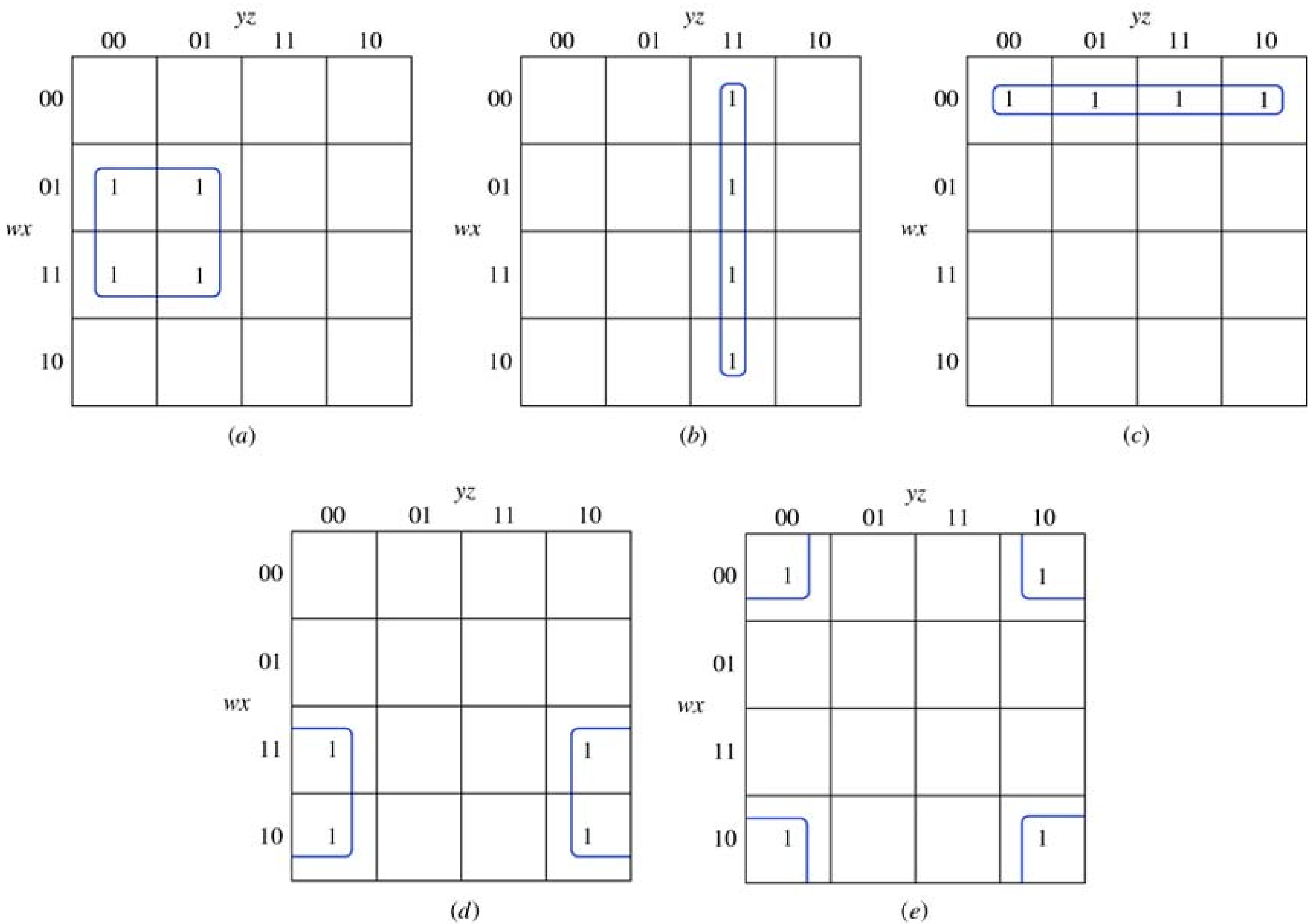
Example of a four-variable map

$wx \backslash yz$	00	01	11	10
00	1	1	0	1
01	1	1	0	0
11	0	0	0	0
10	1	0	0	1

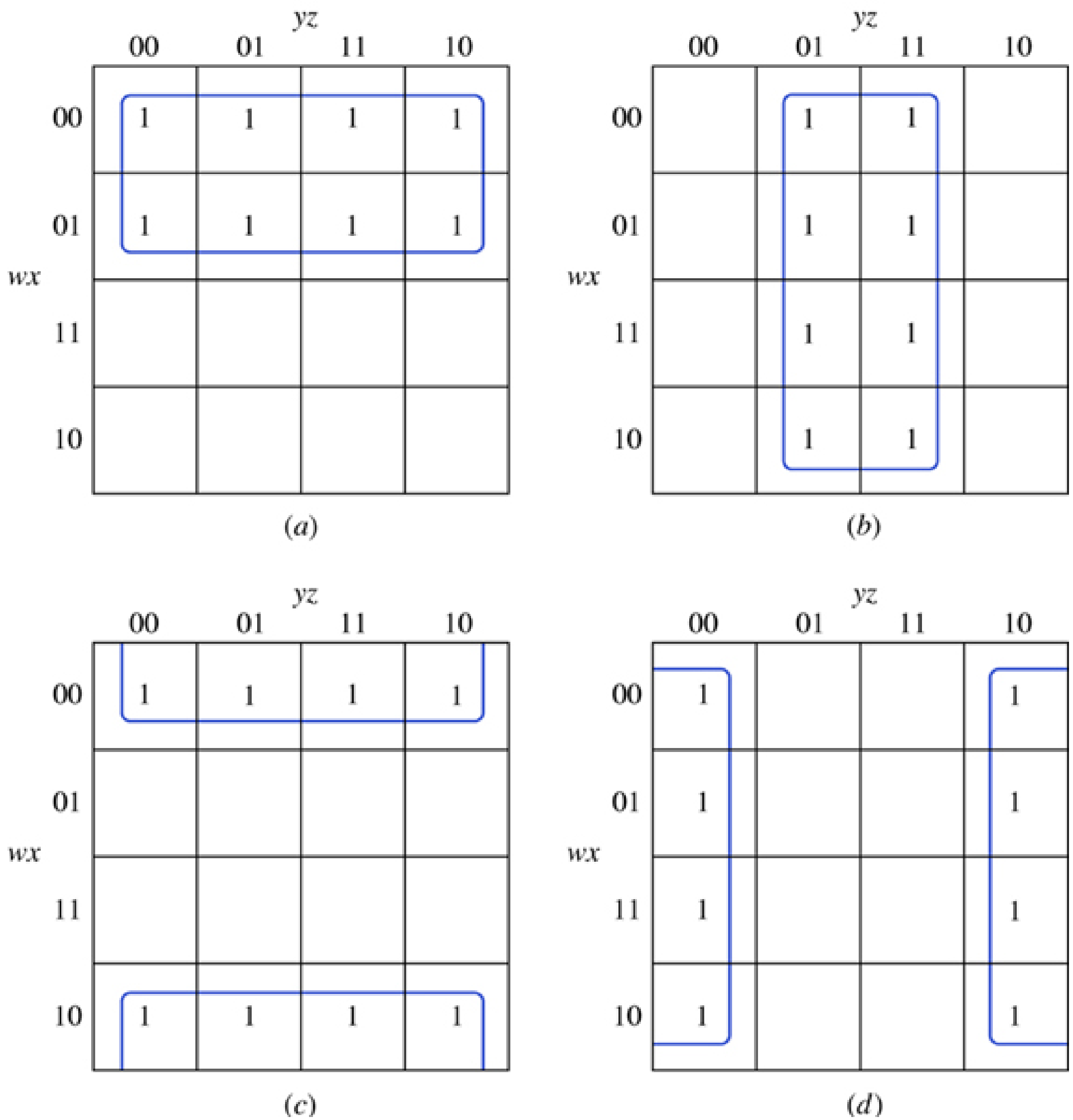
Typical subcubes for elimination of one variable.



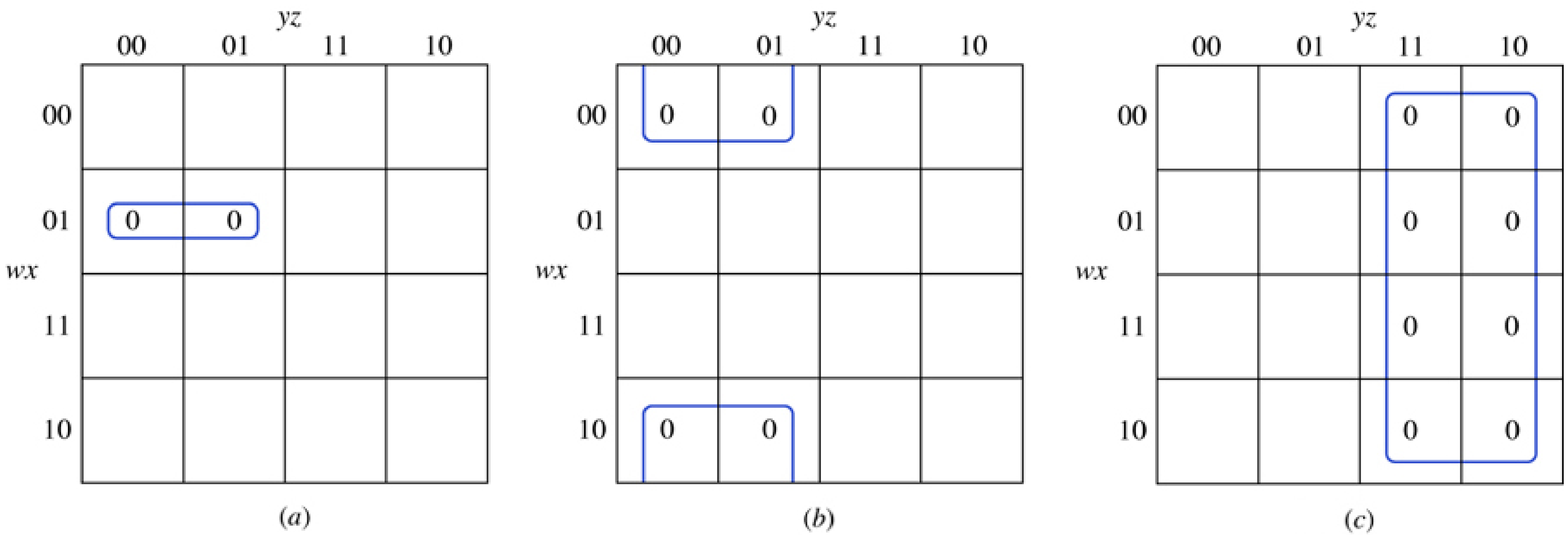
Typical subcubes for elimination of two variables



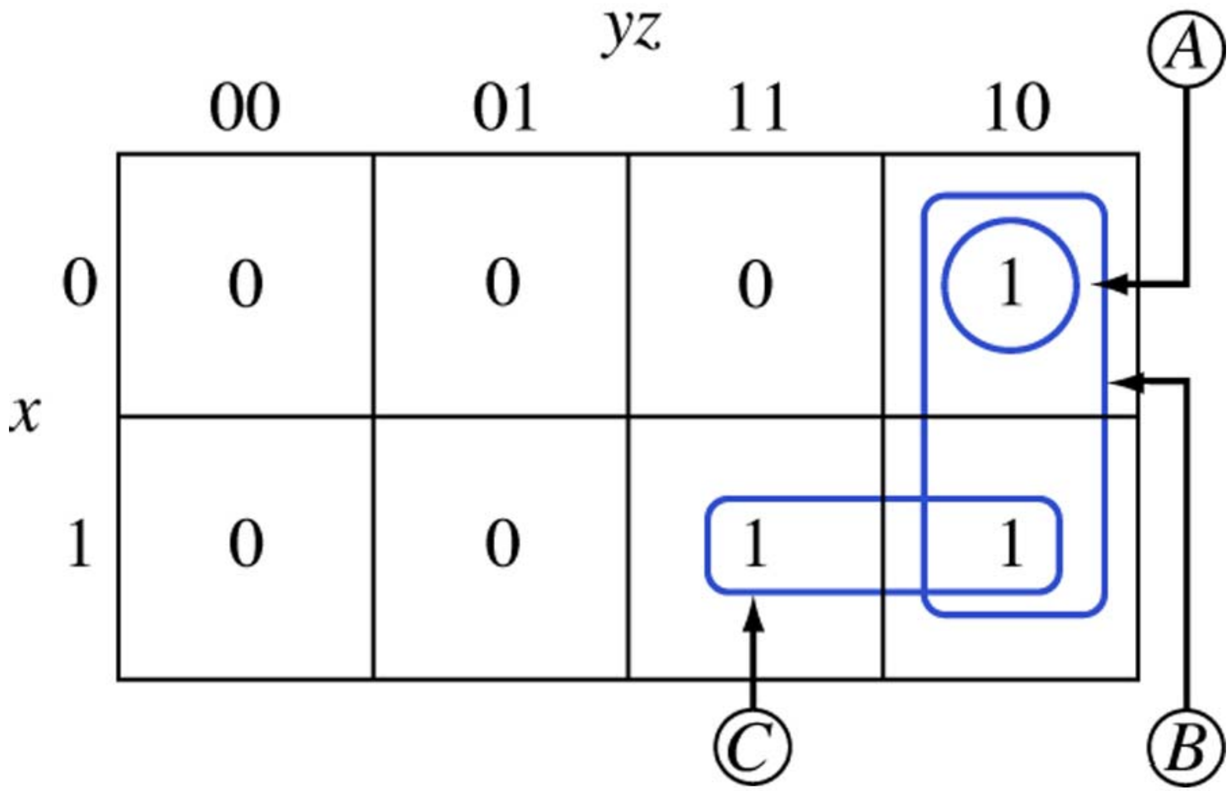
Typical subcubes for elimination of three variables



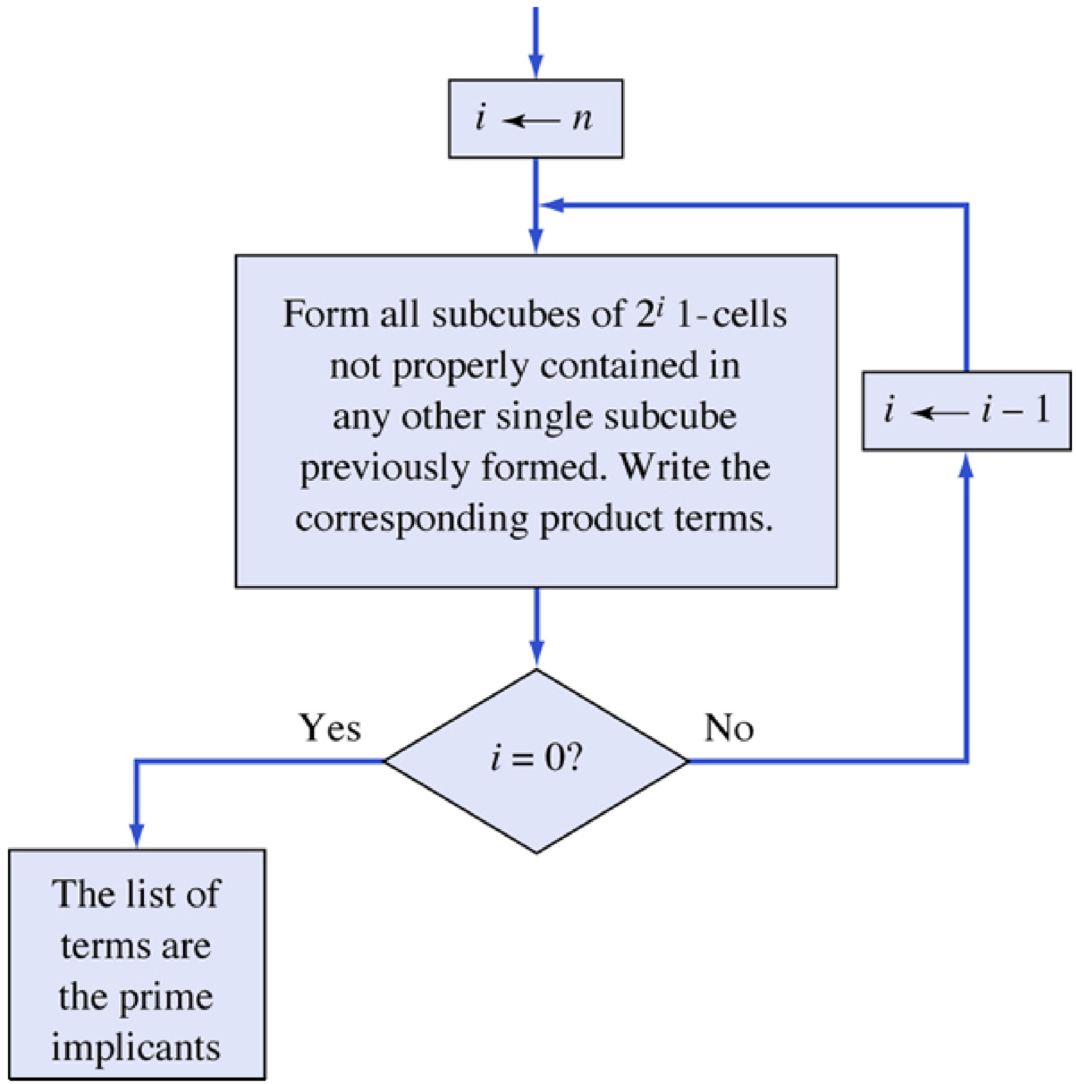
Typical subcubes describing sum terms



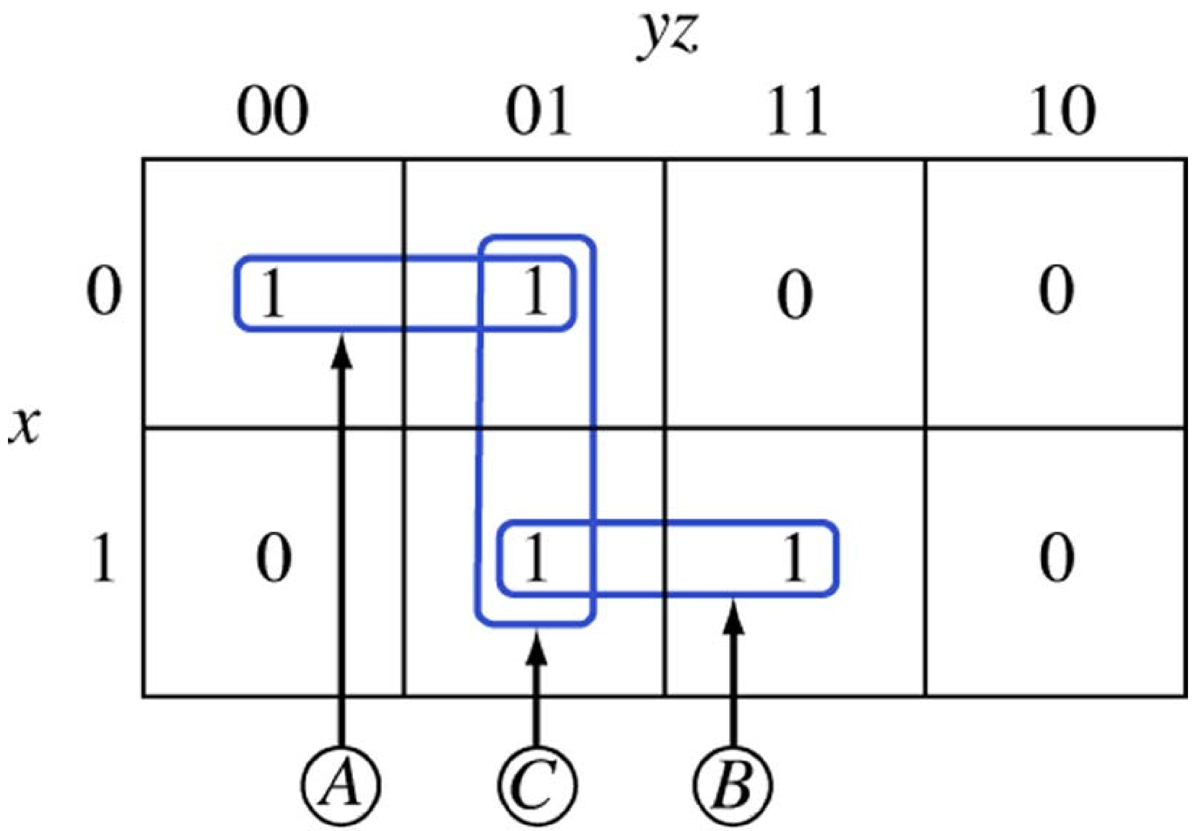
Prime implicants on a map



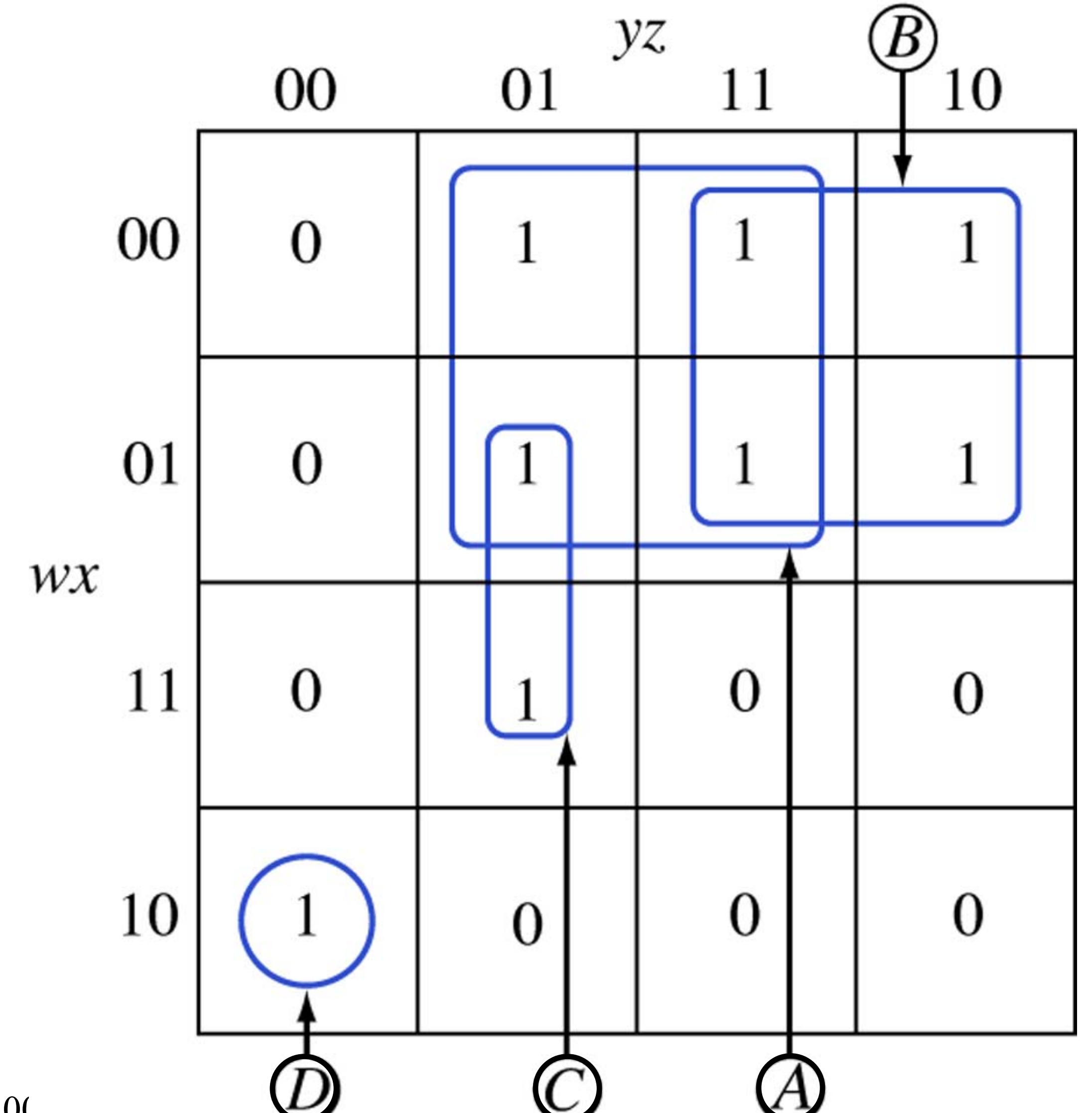
An algorithm for finding all prime implicants



The map for $f(x,y,z) = \Sigma m(0,1,5,7)$



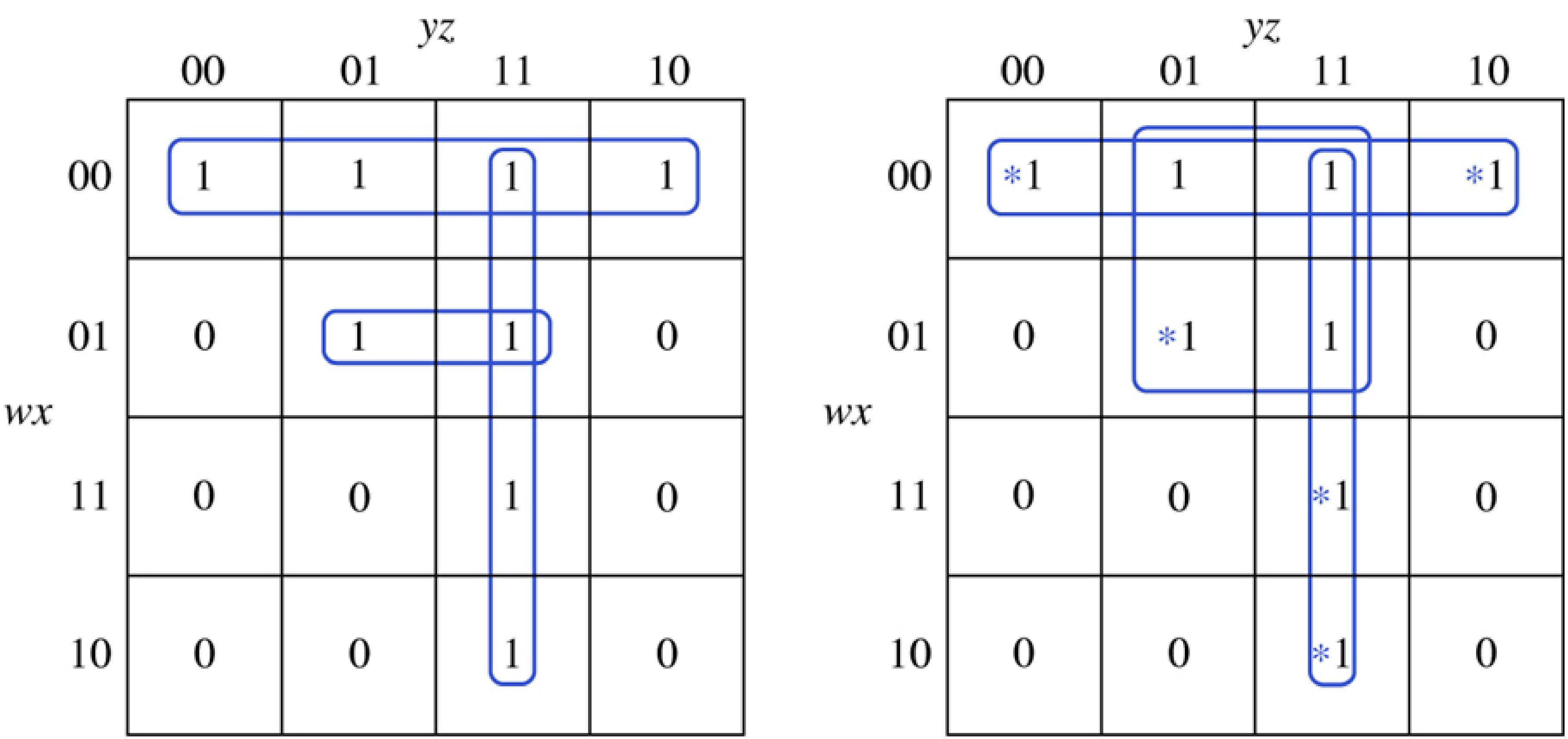
The map for $f(w,x,y,z) = \Sigma m(1,2,3,5,6,7,8,13)$



A quick way to construct the map of f = xy' + wxz + wx'yz

wx \ yz	00	01	11	10
00	0	0	0	0
01	1	1	0	0
11	1	1	1	0
10	0	0	1	0

An example

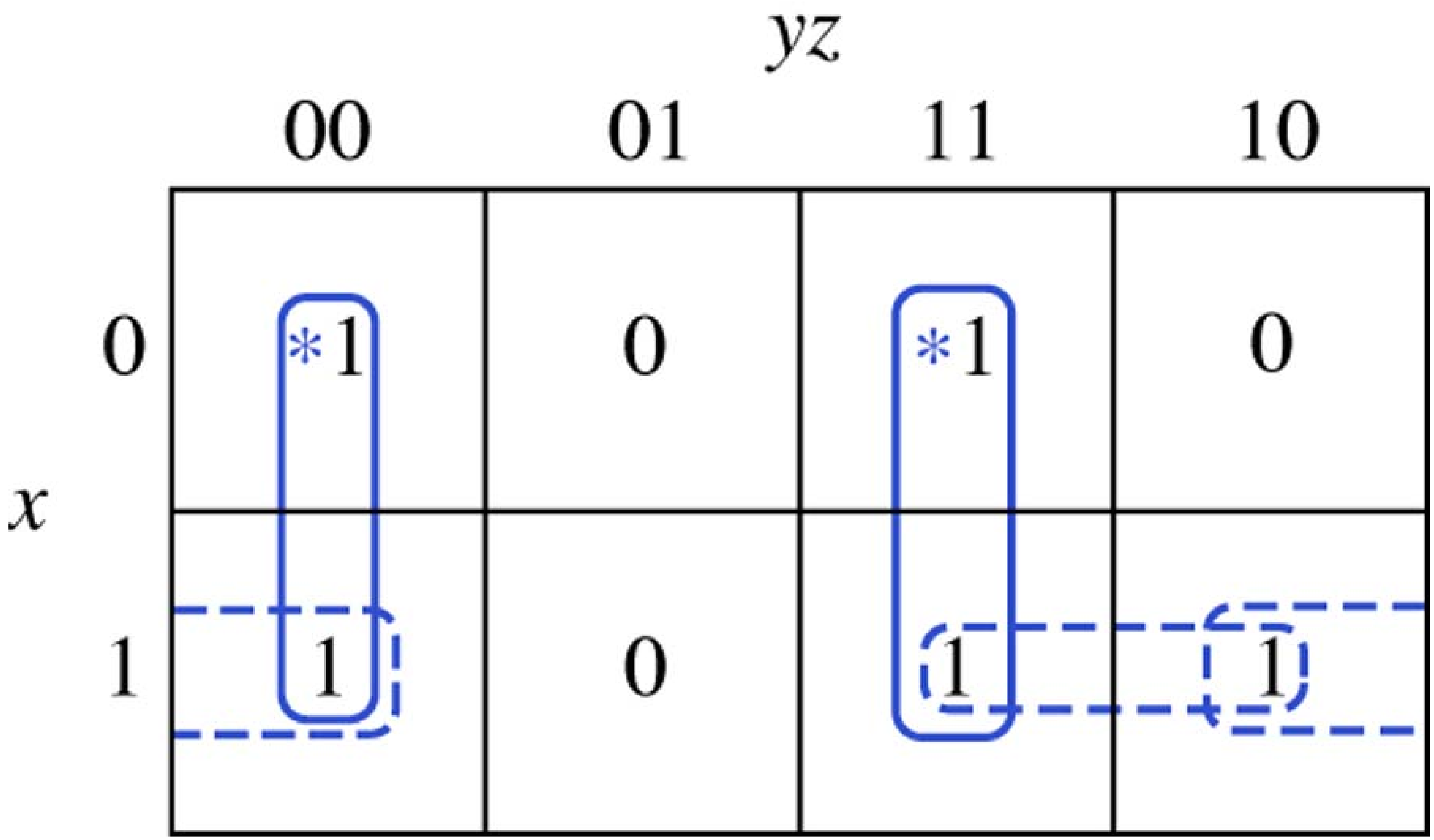


(a) (b)

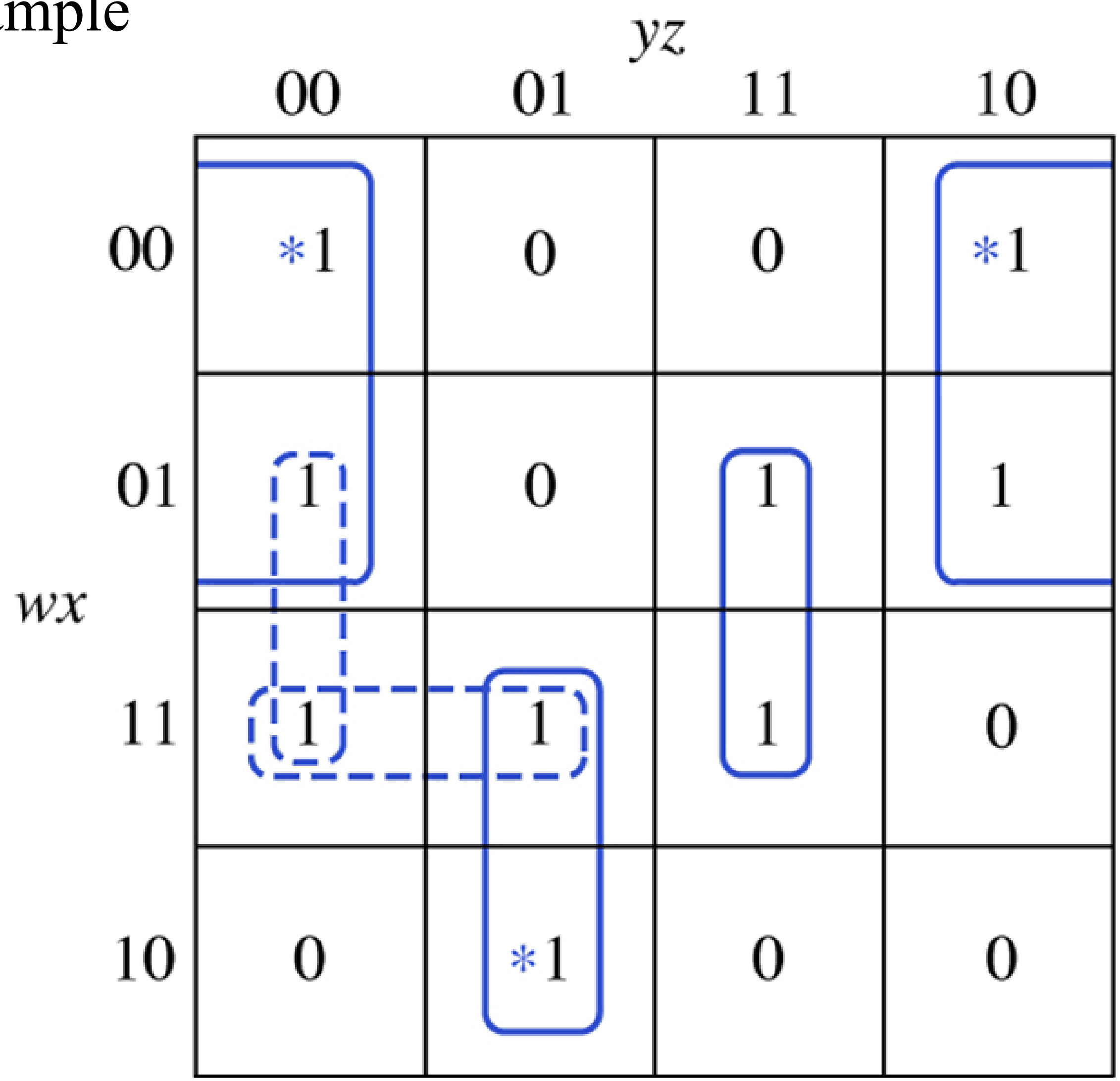
An example

wx \ yz	00	01	11	10
00	1	1	0	*1
01	1	1	*1	0
11	*1	0	0	0
10	0	*1	0	0

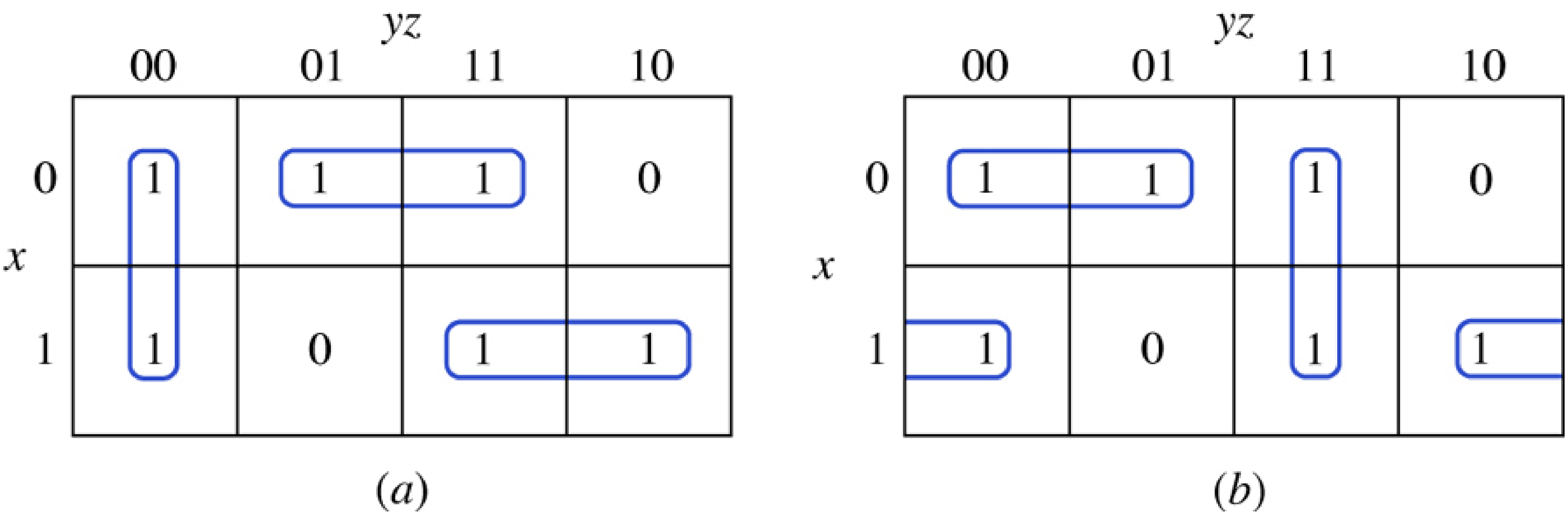
An example



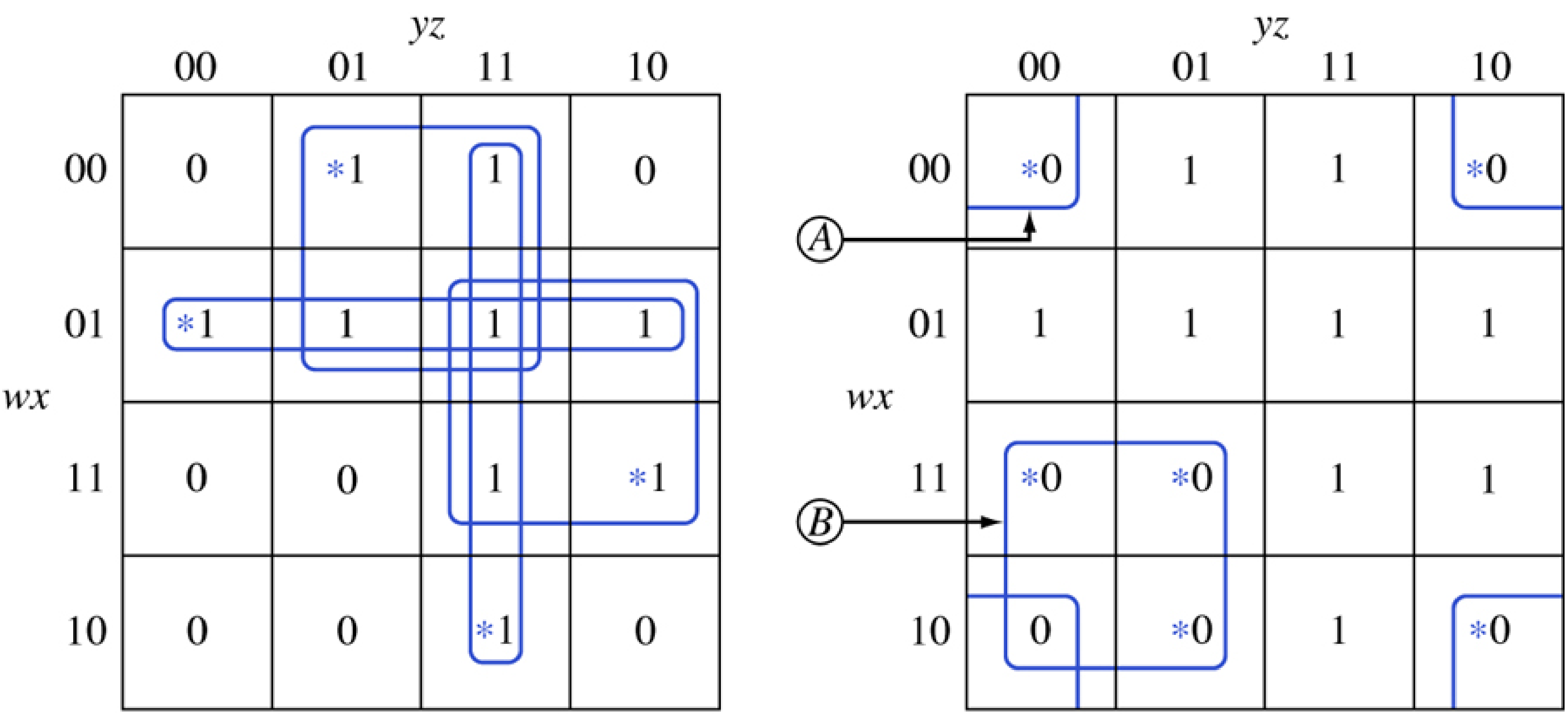
An example



An example



Map for the functions $f(w,x,y,z) = \Sigma m(1,3,4,5,6,7,11,14,15)$



(a) (b)

An example

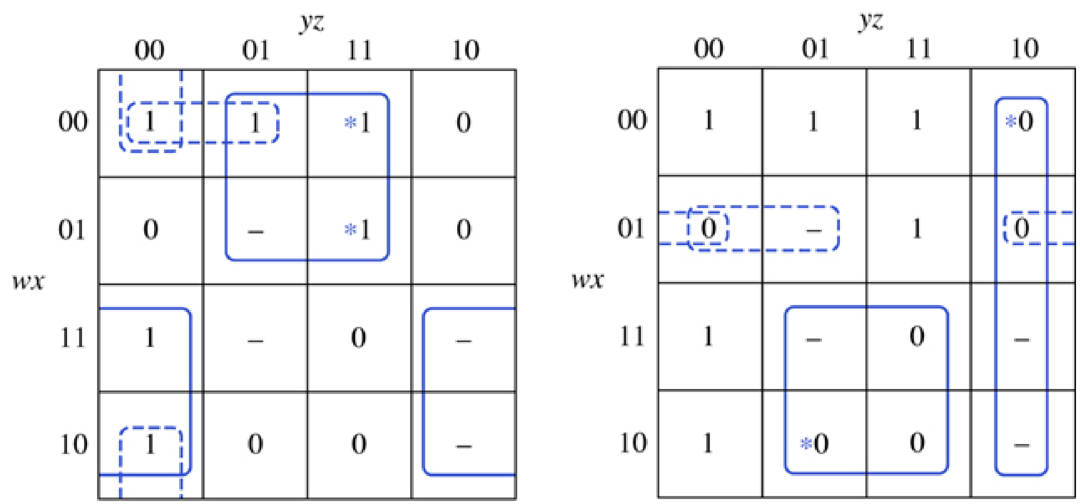
wx \ yz	00	01	11	10
00	1	0	0	1
01	1	*0	1	1
11	1	1	1	*0
10	*0	1	0	0

Incompletely specified Boolean function

$$f(w,x,y,z) = \Sigma m(0,3,7,8,12) + dc(5,10,13,14).$$

w	x	y	z	f
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	–
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	–
1	0	1	1	0
1	1	0	0	1
1	1	0	1	–
1	1	1	0	–
1	1	1	1	0

(*a*)



(*b*) (*c*)

Five-variable Karnaugh maps. (*a*) Reflective structure. (*b*) Layer structure.

Figure 4.26

vw \ xyz	000	001	011	010	110	111	101	100
00	0	1	3	2	6	7	5	4
01	8	9	11	10	14	15	13	12
11	24	25	27	26	30	31	29	28
10	16	17	19	18	22	23	21	20

(*a*)

$v = 0$

wx \ yz	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

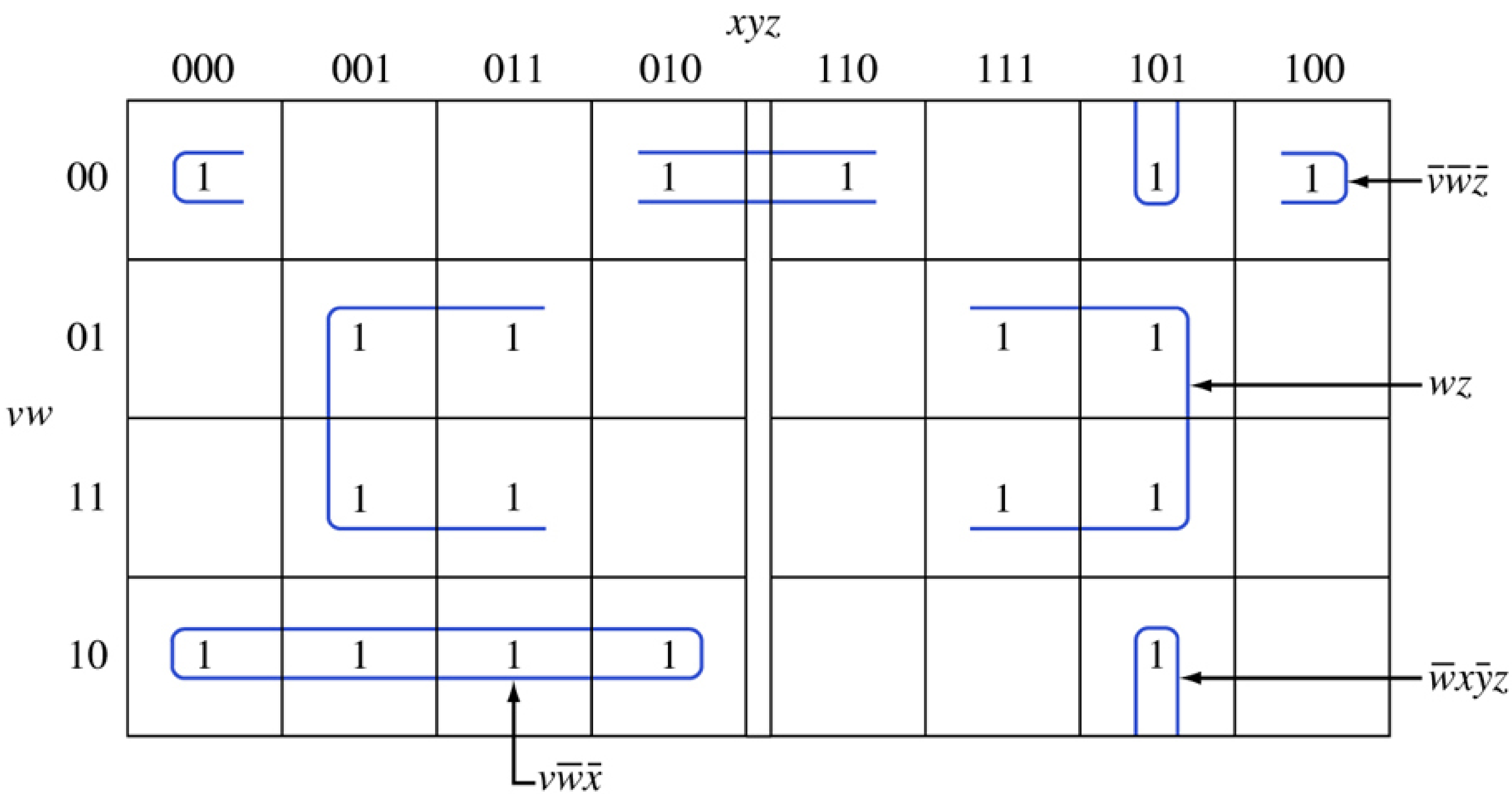
$v = 1$

wx \ yz	00	01	11	10
00	16	17	19	18
01	20	21	23	22
11	28	29	31	30
10	24	25	27	26

(*b*)

Typical subcubes on a five-variable map.

Figure 4.27



Maps for $f(v,w,x,y,z) = \Sigma m(0,1,2,3,6,7,11,15,16,17,19,23,27,31)$.
(a) Subcubes for the minimal sum. (b) Subcubes for the minimal product.

vw \ xyz	000	001	011	010	110	111	101	100
00	1	1	1	1	*1	1	0	0
01	0	0	*1	0	0	*1	0	0
11	0	0	*1	0	0	*1	0	0
10	*1	1	1	0	0	*1	0	0

(a)

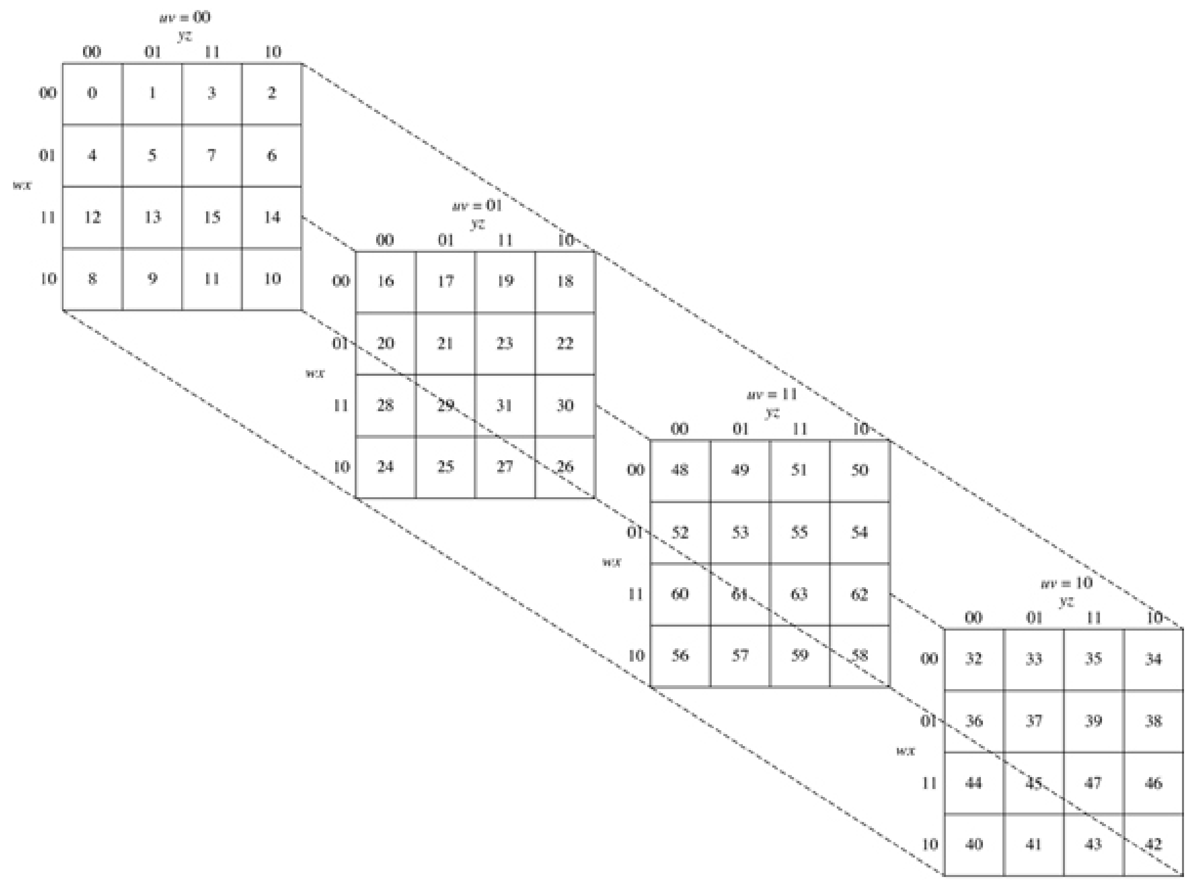
vw \ xyz	000	001	011	010	110	111	101	100
00	1	1	1	1	1	1	*0	*0
01	0	*0	1	*0	*0	1	0	0
11	0	*0	1	0	0	1	0	0
10	1	1	1	*0	0	1	*0	0

(b)

Six-variable Karnaugh maps. (*a*) Reflective structure.
(*b*) Layer structure.

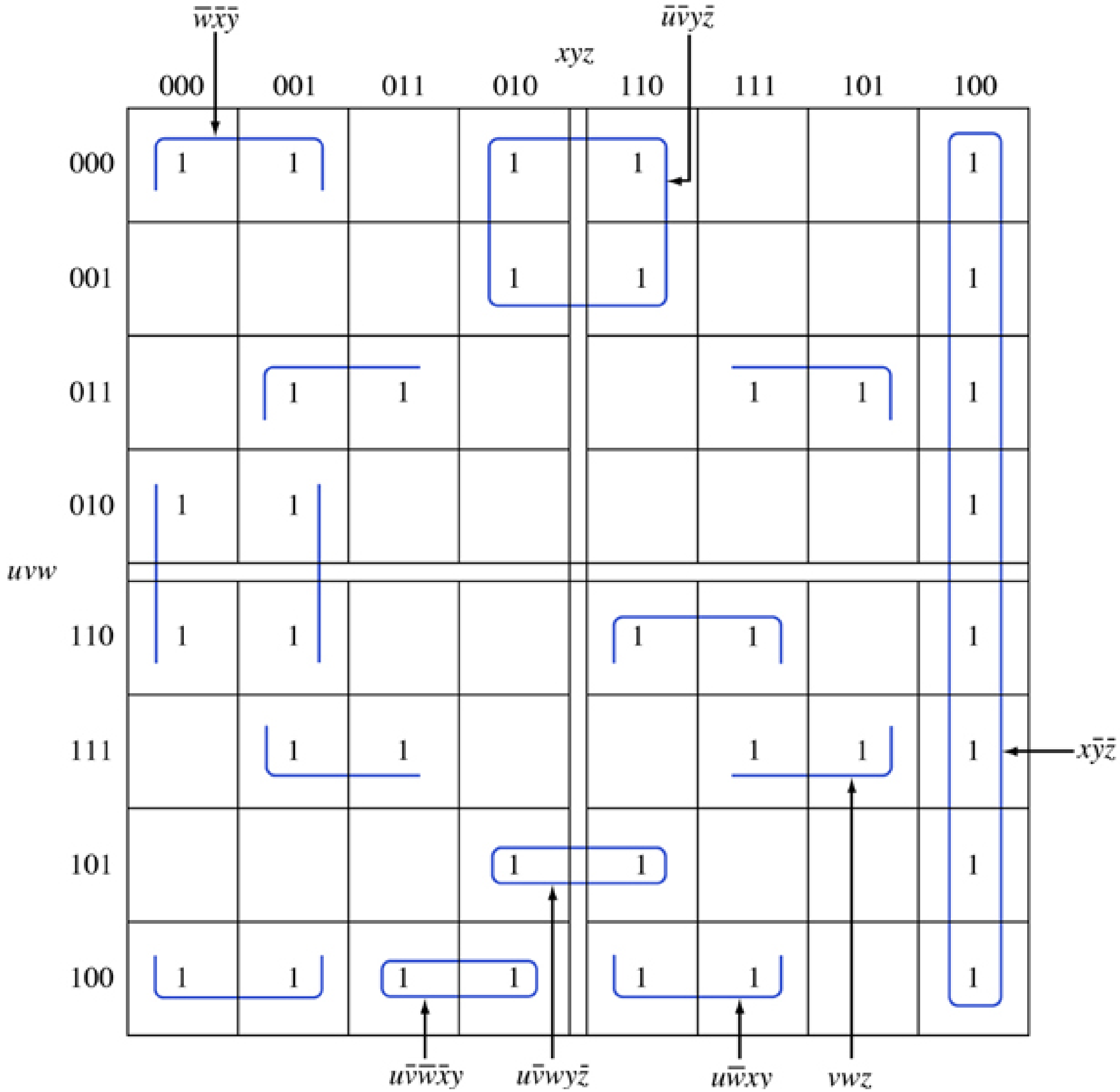
uvw \ xyz	000	001	011	010	110	111	101	100
000	0	1	3	2	6	7	5	4
001	8	9	11	10	14	15	13	12
011	24	25	27	26	30	31	29	28
010	16	17	19	18	22	23	21	20
110	48	49	51	50	54	55	53	52
111	56	57	59	58	62	63	61	60
101	40	41	43	42	46	47	45	44
100	32	33	35	34	38	39	37	36

(*a*)

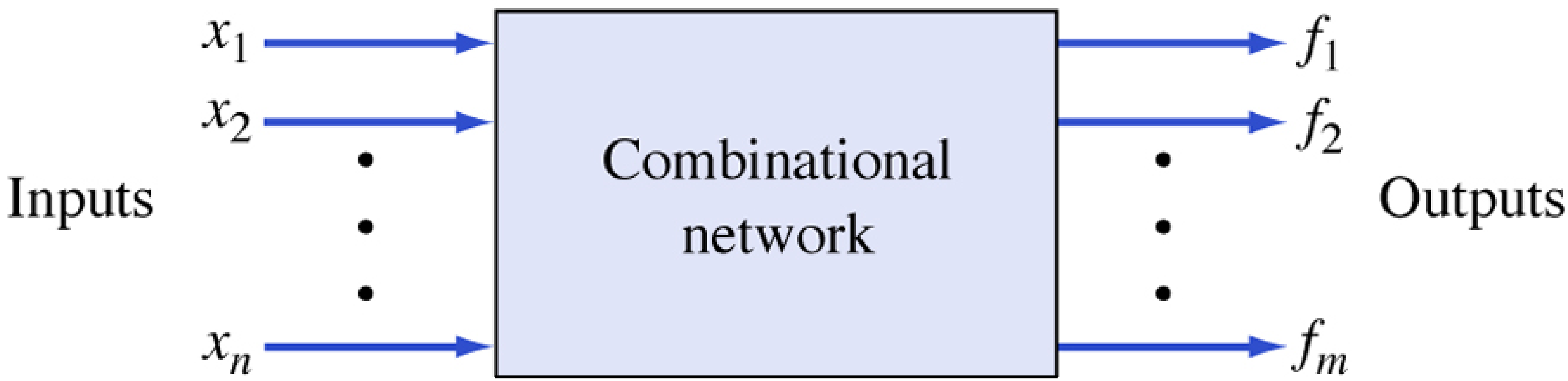


(*b*)

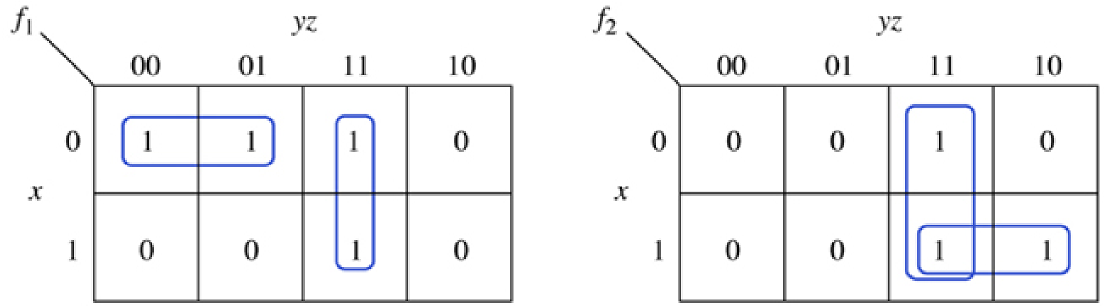
Typical subcubes on a six-variable map



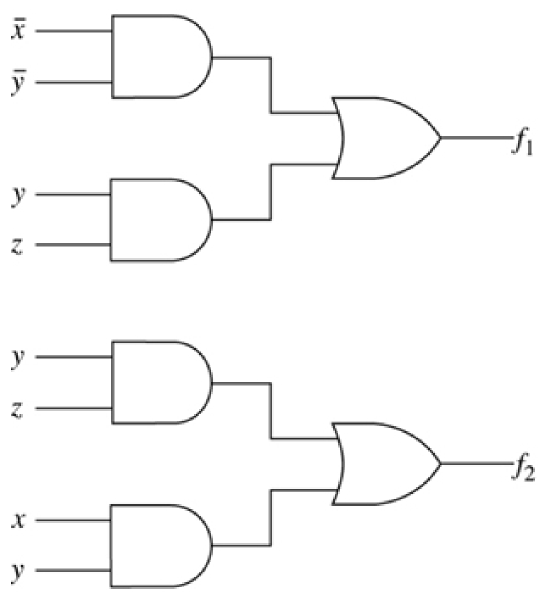
General form of a combinational network



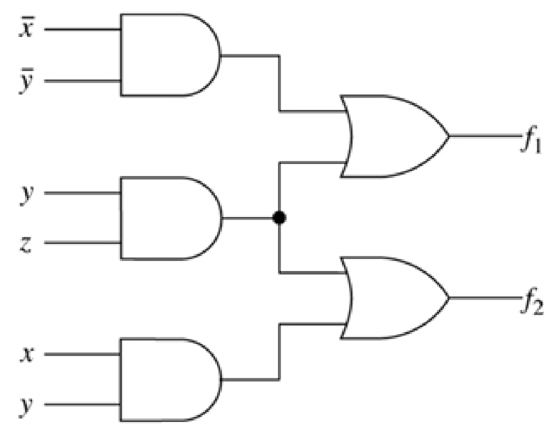
Minimization through sharing in a multiple output network



(a)

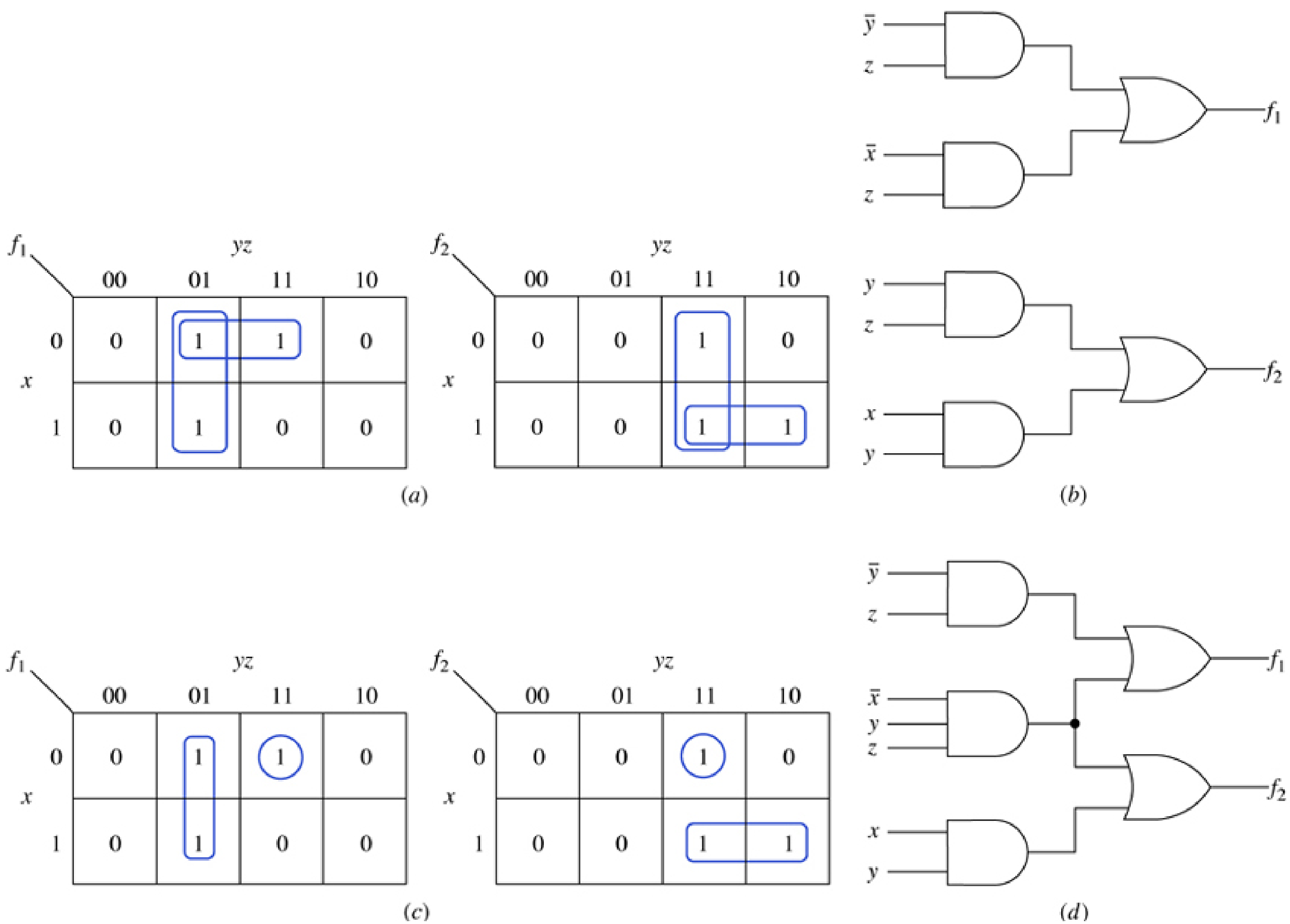


(b)



(c)

Minimization through sharing: optimization of individual output does not necessarily lead to overall optimization



Map compression of a three-variable function

x	y	z	f
0	0	0	f_0
0	0	1	f_1
0	1	0	f_2
0	1	1	f_3
1	0	0	f_4
1	0	1	f_5
1	1	0	f_6
1	1	1	f_7

(*a*)

x \ yz	00	01	11	10
0	f_0	f_1	f_3	f_2
1	f_4	f_5	f_7	f_6

(*b*)

x \ y	0	1
0	$f_0 \cdot \bar{z} + f_1 \cdot z$	$f_2 \cdot \bar{z} + f_3 \cdot z$
1	$f_4 \cdot \bar{z} + f_5 \cdot z$	$f_6 \cdot \bar{z} + f_7 \cdot z$

(*c*)

x \ z	0	1
0	$f_0 \cdot \bar{y} + f_2 \cdot y$	$f_1 \cdot \bar{y} + f_3 \cdot y$
1	$f_4 \cdot \bar{y} + f_6 \cdot y$	$f_5 \cdot \bar{y} + f_7 \cdot y$

(*d*)

y \ z	0	1
0	$f_0 \cdot \bar{x} + f_4 \cdot x$	$f_1 \cdot \bar{x} + f_5 \cdot x$
1	$f_2 \cdot \bar{x} + f_6 \cdot x$	$f_3 \cdot \bar{x} + f_7 \cdot x$

(*e*)

x: 0	x: 1
$f_0 \cdot \bar{y}\bar{z} + f_1 \cdot \bar{y}z + f_2 \cdot y\bar{z} + f_3 \cdot yz$	$f_4 \cdot \bar{y}\bar{z} + f_5 \cdot \bar{y}z + f_6 \cdot y\bar{z} + f_7 \cdot yz$

(*f*)

Example of a variable-entered map.

x	y	z	f
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

(a)

$x \backslash y$	0	1
0	1	$\bar{z}$
1	z	0

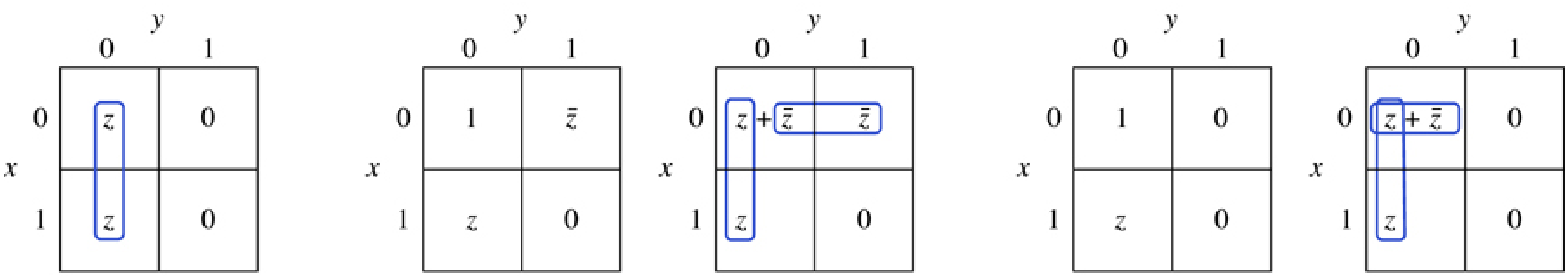
(b)

Example of a variable-entered map with infrequently appearing variables.

$x \backslash yz$	00	01	11	10
0	A	1	1	0
1	0	0	1	B

Variable-entered maps grouping techniques

$\bar{z}$



(a) (b) (c)

Minimization through a map with single-variable entries

w \ xy	00	01	11	10
0	z	1	1	0
1	0	$\bar{z}$	1	0

(*a*)

w \ xy	00	01	11	10
0	z	1	1	0
1	0	0	1	0

w \ xy	00	01	11	10
0	0	1	1	0
1	0	$\bar{z}$	1	0

(*b*)

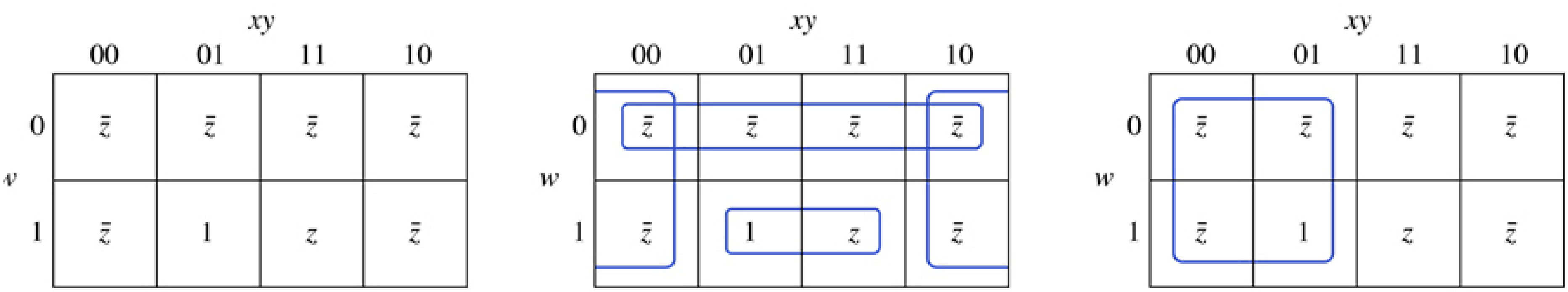
w \ xy	00	01	11	10
0	0	–	1	0
1	0	0	1	0

(*c*)

Optimal groupings on a variable-entered map

$w \backslash xy$	00	01	11	10
0	z	0	z	1
1	z	$\bar{z}$	1	0

Minimization through a map with single-variable entries



(a)

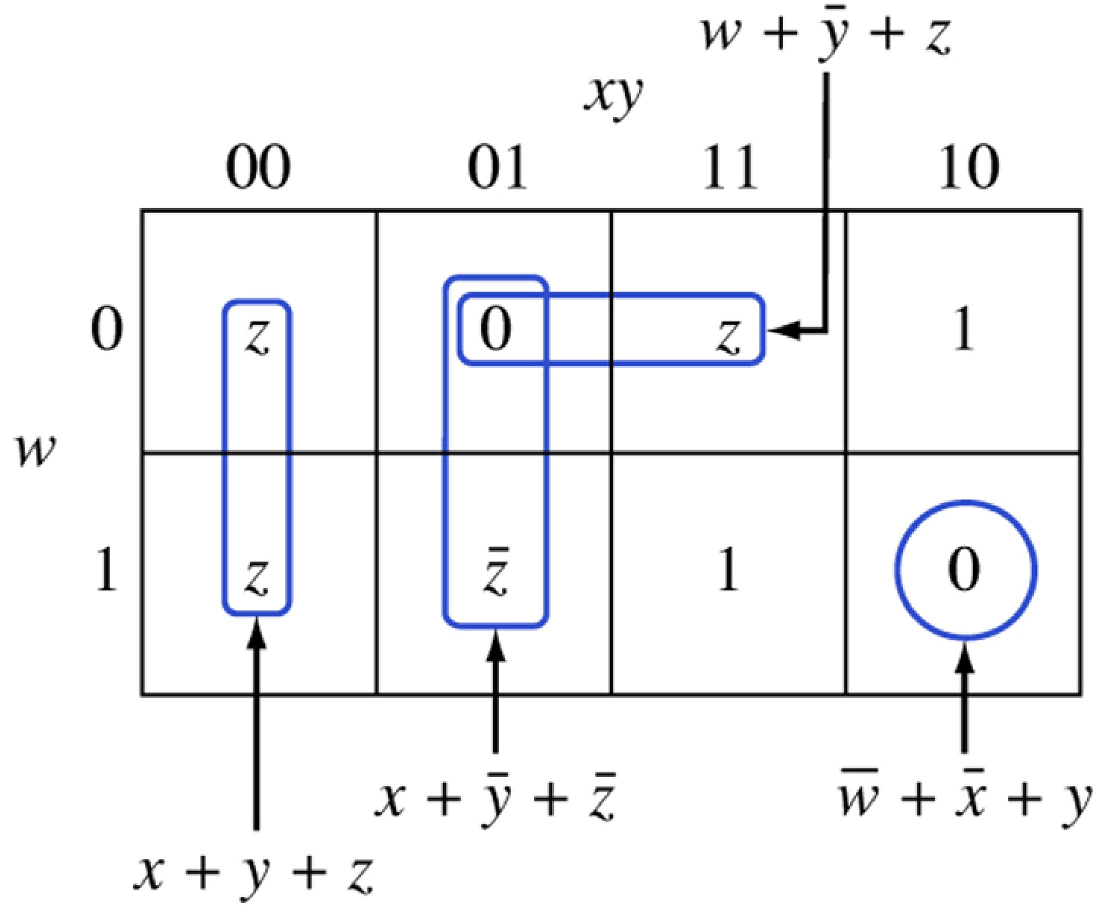
$f(w,x,y,z) = \overline{w}\bar{z} + \bar{y}\bar{z} + wyz + \cdots$

(b)

$f(w,x,y,z) = \overline{w}\bar{z} + \bar{y}\bar{z} + wyz + \bar{x}\bar{z}$

((c)

Minimization through a map with single-variable entries



Obtaining a minimal sum for the incompletely specified Boolean funciton $f(w,x,y,z) = \Sigma m(3,5,6,7,8,9,10) + dc(4,11,12,14,15)$ using a variable-entered map. (*a*) Truth table. (*b*) Variable-entered map. (*c*) Step 1 map and subcubes. (*d*) Step 2 map and subcubes.

Figure 4.42

w	x	y	z	f	$f_i \cdot \bar{z} + f_j \cdot z$	Map entry
0	0	0	0	0	$0 + 0$	0
0	0	0	1	0		
0	0	1	0	0	$0 + z$	z
0	0	1	1	1		
0	1	0	0	–	$- \cdot \bar{z} + z$	z,1
0	1	0	1	1		
0	1	1	0	1	$\bar{z} + z$	1
0	1	1	1	1		
1	0	0	0	1	$\bar{z} + z$	1
1	0	0	1	1		
1	0	1	0	1	$\bar{z} + - \cdot z$	$\bar{z}$,1
1	0	1	1	–		
1	1	0	0	–	$- \cdot \bar{z} + 0$	$\bar{z}$,0
1	1	0	1	0		
1	1	1	0	–	$- \cdot \bar{z} + - \cdot z$	–
1	1	1	1	–		

(*a*)

w \ xy	00	01	11	10
0	0	z	1	z,1
1	1	$\bar{z}$,1	–	$\bar{z}$,0

(*b*)

w \ xy	00	01	11	10
0	0	z	1	z,1
1	1	$\bar{z}$,1	–	$\bar{z}$,0

This cell is being used as a 1-cell.

(*c*)

w \ xy	00	01	11	10
0	0	0	1	1
1	1	1	–	0

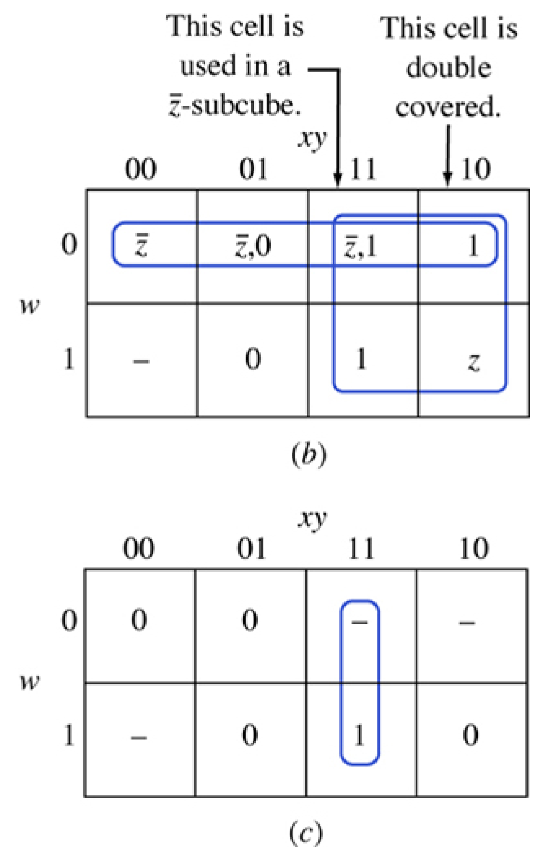
(*d*)

Obtaining a minimal sum for the incompletely specified Boolean function $f(w,x,y,z) = \Sigma m(0,4,5,6,13,14,15) + dc(2,7,8,9)$ using a variable-entered map. (*a*) Truth table.
(*b*) Step 1 map and subcubes. (*c*) Step 2 map and subcubes.

Figure 4.43

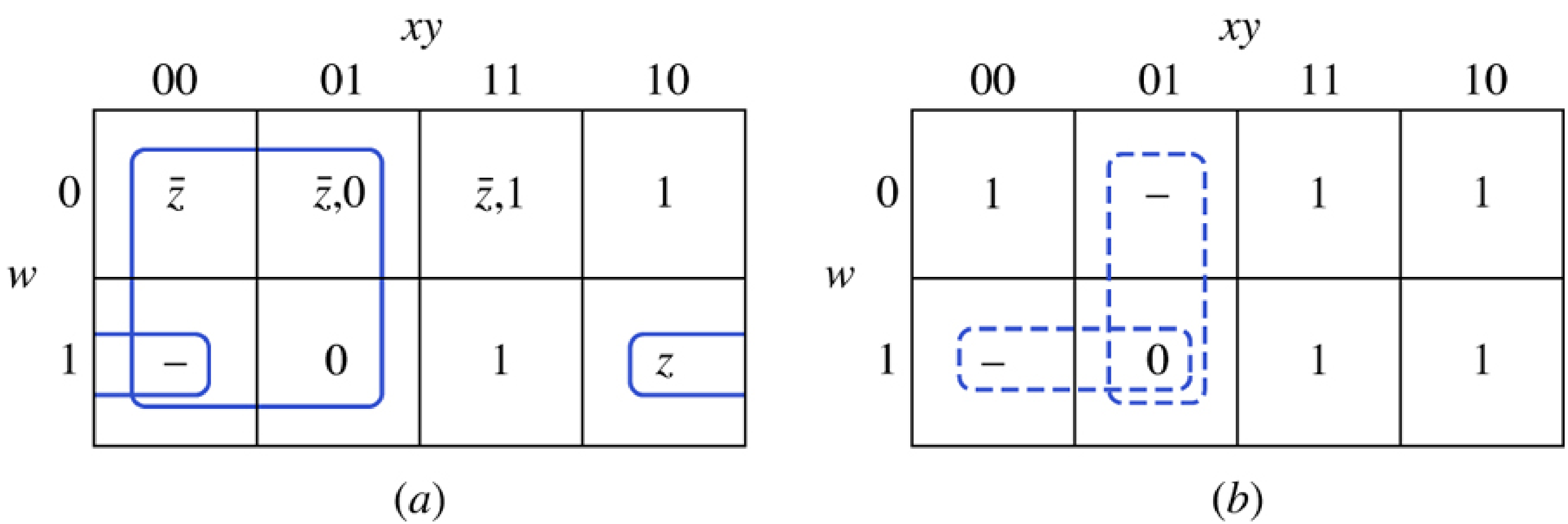
w	x	y	z	f	$f_i \cdot \bar{z} + f_j \cdot z$	Map entry
0	0	0	0	1	$\bar{z} + 0$	$\bar{z}$
0	0	0	1	0		
0	0	1	0	–	$- \cdot \bar{z} + 0$	$\bar{z}$,0
0	0	1	1	0		
0	1	0	0	1	$\bar{z} + z$	1
0	1	0	1	1		
0	1	1	0	1	$\bar{z} + - \cdot z$	$\bar{z}$,1
0	1	1	1	–		
1	0	0	0	–	$- \cdot \bar{z} + - \cdot z$	–
1	0	0	1	–		
1	0	1	0	0	$0 + 0$	0
1	0	1	1	0		
1	1	0	0	0	$0 + z$	z
1	1	0	1	1		
1	1	1	0	1	$\bar{z} + z$	1
1	1	1	1	1		

(*a*)



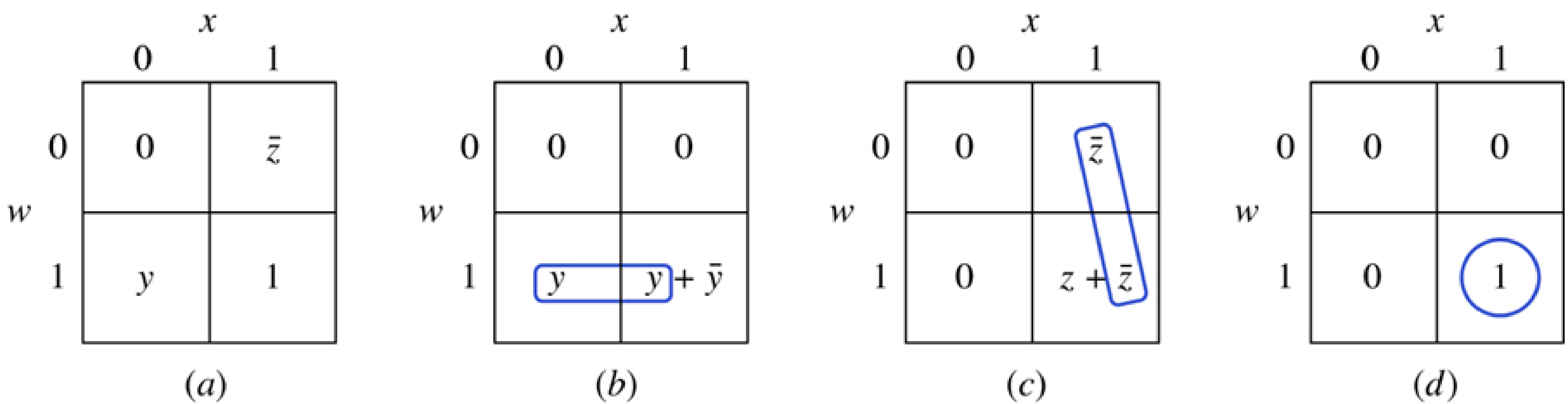
Obtaining a minimal product for the incompletely specified Boolean function $f(w,x,y,z) = \Sigma m(0,4,5,6,13,14,15) + dc(2,7,8,9)$ using a variable-entered map. (*a*) Step 1 map and subcubes. (*b*) Step 2 map and subcubes.

Figure 4.44



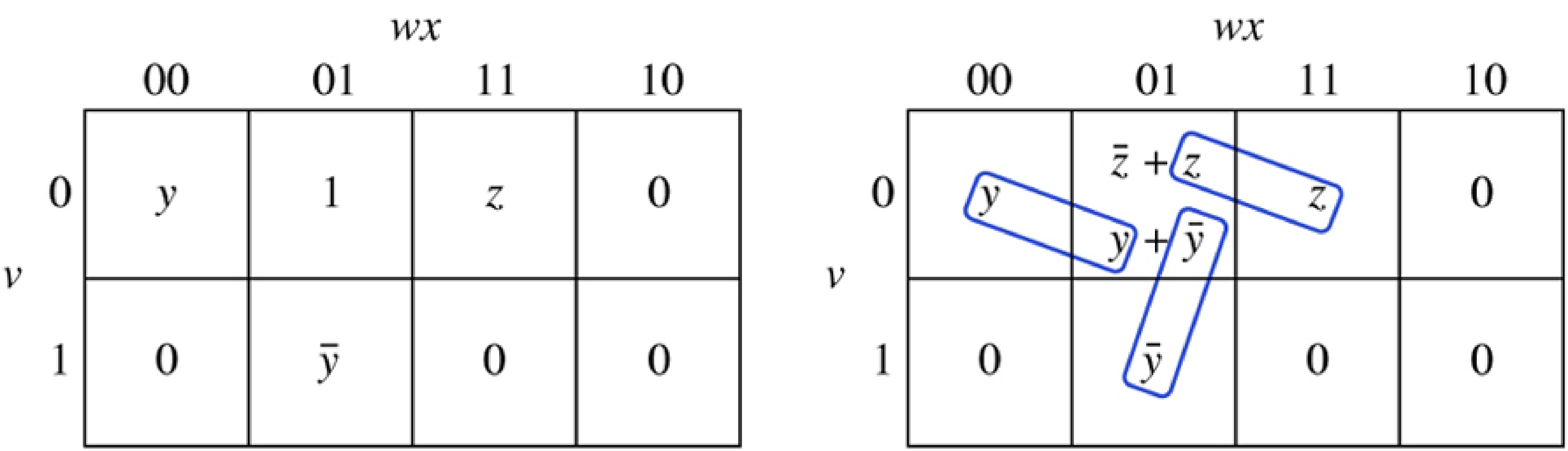
Maps having entries involving more than one variable. (*a*) Variable-entered map. (*b*) Grouping the *y* literal. (*c*) Grouping the $\bar{z}$ literal. (*d*) Grouping the not completely covered 1-cell.

Figure 4.45



Obtaining a minimal sum from a variable-entered map having several single-literal map entries. (*a*) Variable-entered map. (*b*) Optimal collection of subcubes.

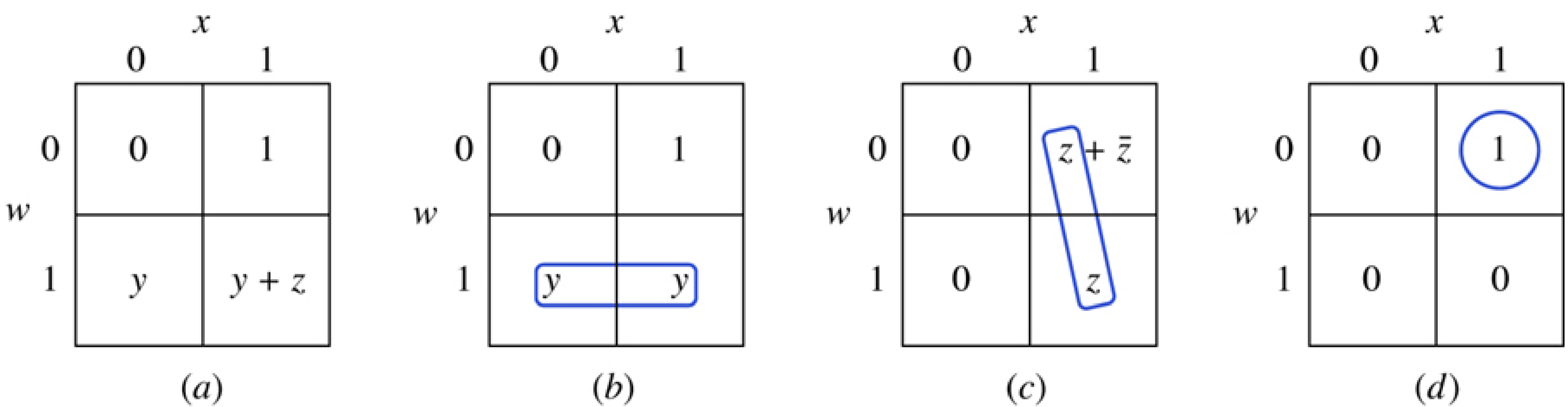
Figure 4.46



(*a*) (*b*)

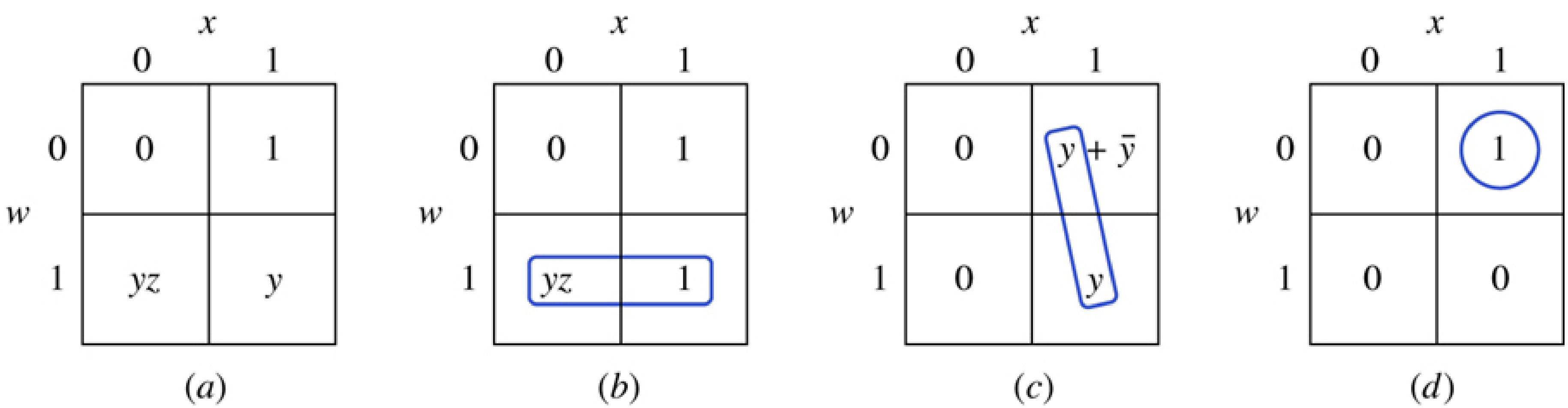
Maps having sum terms as entries. (*a*) Variable-entered map. (*b*) Grouping the *y* literal. (*c*) Grouping the *z* literal. (*d*) Grouping the not completely covered 1-cell.

Figure 4.47



Maps having product terms as entries. (*a*) Variable-entered map. (*b*) Grouping the *yz* term. (*c*) Grouping the *y* literal. (*d*) Grouping the not completely covered 1-cell.

Figure 4.48



Maps having product and sum terms as entries.
(*a*) Variable-entered map. (*b*) Grouping the *yz* term.
(*c*) Grouping the $\bar{y}$ literal.

Figure 4.49